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ABSTRACT

This is part three of a three-part School Mathematics Study Group (SMSG) textbook. It contains eight chapters of supplements. Four of these are designed to be supplements to the specific chapters 2, 3, 7, and 10. The other four supplements cover the topics: Determinants; Flow Chart for Two Linear Equations in x and y; Graphs with Non-uniform Scales; and Points, Lines, and Planes, which can be used as a supplement to Chapters 2, 3, and 8. (MK)

SCHOOL
MATHEMATICS
STUDY GROUP

ANALYTIC GEOMETRY

Student Text

Part 3

(revised edition)

SMSG

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Chapter I

Supplement A

DETERMINANTS

If we suppose that this system of equations has a solution:

$$ax + by = c$$
$$px + qy = r$$

it can be found by elementary methods to be:

$$x = \frac{cq - br}{aq - bp}, \quad y = \frac{ar - cp}{aq - bp}.$$

These numerators and denominators may be written in a form which helps to develop a useful algebraic concept and notation:

$$x = \frac{\begin{vmatrix} c & b \\ r & q \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & c \\ p & r \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}}$$

An expression of the form $\begin{vmatrix} a & b \\ p & q \end{vmatrix}$ is called a determinant, and its value, as suggested by the example above, is defined:

$$\begin{vmatrix} a & b \\ p & q \end{vmatrix} = aq - bp$$

This determinant has two rows: a,b; and p,q; and two columns: a,p and b,q. It is called a second order determinant, and has $4 = 2^2$ terms or elements. A third order determinant has three rows and three columns, and $9 = 3^2$ elements. A determinant of order n has n rows and n columns, and so on. We frequently use "Δ" to indicate either a determinant or its value. Note that the first order determinant $|a|$ has the value a.

We list a number of theorems, all of which are true for determinants of any order, and indicate briefly proofs for the second order. In most cases the proof for higher orders is a straightforward generalization of the proof for the second order.

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THEOREM 1. Δ is unchanged if we interchange rows with columns.

$$\begin{vmatrix} a & b \\ p & q \end{vmatrix} = \begin{vmatrix} a & p \\ b & q \end{vmatrix} = aq - bp = \Delta$$

Note: All these theorems remain valid if we interchange the words "row", "column."

THEOREM 2. If two rows of Δ are interchanged, the sign of Δ is changed.

$$\begin{vmatrix} p & q \\ a & b \end{vmatrix} = bp - aq = -(aq - bp) = -\Delta .$$

THEOREM 3. If every element of a row of Δ is multiplied by k, then so is Δ.

$$\begin{vmatrix} ka & kb \\ p & q \end{vmatrix} = kaq - kbp = k(aq - bp) = k\Delta .$$

THEOREM 4. If two rows of Δ are equal or proportional, then $\Delta = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0 , \quad \begin{vmatrix} a & b \\ ka & kb \end{vmatrix} = k\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 .$$

THEOREM 5. Two determinants may be added if they agree in all the elements of $n - 1$ rows. Their sum is then a determinant with these same $n - 1$ rows; and the elements of the remaining row are the sums of the corresponding elements in the original determinants.

$$\begin{vmatrix} a & b \\ p & q \end{vmatrix} + \begin{vmatrix} c & d \\ p & q \end{vmatrix} = aq - bp + cq - dp = (a + c)q - (b + d)p$$

$$= \begin{vmatrix} a + c & b + d \\ p & q \end{vmatrix} .$$

THEOREM 6. A determinant is unchanged if, to the elements of any row we add a common multiple of the corresponding elements of another row.

$$\begin{vmatrix} a + kp & b + kq \\ p & q \end{vmatrix} = \begin{vmatrix} a & b \\ p & q \end{vmatrix} + \begin{vmatrix} kp & kq \\ p & q \end{vmatrix} = \Delta + 0 = \Delta .$$

Notation. It is convenient, for purposes of generalization, to use "double subscript notation."

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

where "a_{ij}" designates the element in row i and column j.

Exercise. Rewirte the proofs of Theorems 1-6 using double subscript notation.

DEFINITION. Minor of a_{ij} (Notation A_{ij}) is the determinant of the square array obtained by removing from Δ all elements of row i, and of column j; we sometimes use the same word to indicate the value of that determinant. Note that A_{ij} is of order $n - 1$.

DEFINITION. Cofactor of a_{ij} (Notation α_{ij}) $\alpha_{ij} = (-1)^{i+j} A_{ij}$. Note that α_{ij} is the same as A_{ij} if the sum of its row and column numbers is even, and α_{ij} is the negative of A_{ij} if the sum of its row and column numbers is odd. As above, we use "cofactor" to indicate the expression as well as its value.

Example 1.

$$\begin{vmatrix} a & b \\ p & q \end{vmatrix}$$

The minor of a is q; of p is b.
The cofactor of a is q; of p is $-b$.

Example 2.

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$$

The minor of p is $\begin{vmatrix} b & c \\ v & w \end{vmatrix}$, of c is $\begin{vmatrix} p & q \\ u & v \end{vmatrix}$.

The cofactor of p is $(-1)^{2+1} \begin{vmatrix} b & c \\ v & w \end{vmatrix} = -\begin{vmatrix} b & c \\ v & w \end{vmatrix}$.

The cofactor of c is $(-1)^{1+3} \begin{vmatrix} p & q \\ u & v \end{vmatrix} = \begin{vmatrix} p & q \\ u & v \end{vmatrix}$.

Example 3.

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{vmatrix}$$

The minor of 8 is $\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}$ which has the value $21 - 24 = -3$.

The cofactor of 8 is $(-1)^{3+1}$ times the minor of 8, and also has the value -3.

The minor of 9 is $\begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix}$ which has the value $14 - 20 = -6$.

The cofactor of 9 is $(-1)^{3+2}$ times the minor of 9, and has the value 6.

Exercise. Find the cofactors of each of the nine elements of (3) above, or by applying Theorem 6 to write the determinant in a form simpler to evaluate, thus:

(1) Write the same second column, then add (-2) times these elements to the corresponding element of the third column; then add (-4) times these same elements to the corresponding element of the first column:

$$\begin{vmatrix} 4(-4)+3 & 4 & 4(-2)+1 \\ 3(-4)+2 & 3 & 3(-2)+5 \\ 1(-4)+4 & 1 & 1(-2)+2 \end{vmatrix}$$

which yields the equal determinant

$$\begin{vmatrix} -13 & 4 & -7 \\ -10 & 3 & -1 \\ 0 & 1 & 0 \end{vmatrix}.$$

If we now evaluate by using the element of the third row, we get

$$0\begin{vmatrix} 4 & -7 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} -13 & -7 \\ -10 & -1 \end{vmatrix} + 0\begin{vmatrix} -13 & 4 \\ -10 & 3 \end{vmatrix} = 0 - 1(13 - 70) + 0 = -1(-57) = 57$$

DEFINITION. The value of any determinant is equal to the sum of the products of the elements of the first row by their corresponding cofactors. Application: Cramer's Rule?

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a|d| - b|c| = ad - bc.$$

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = a\begin{vmatrix} q & r \\ v & w \end{vmatrix} - b\begin{vmatrix} p & r \\ u & w \end{vmatrix} + c\begin{vmatrix} p & q \\ u & v \end{vmatrix}$$

$$= a(qw - rv) - b(pw - ru) + c(pv - qu), \text{ etc.}$$

Example. $$\begin{vmatrix} 3 & 4 & 1 \\ 2 & 3 & 5 \\ 4 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} - 4\begin{vmatrix} 2 & 5 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= 3(6 - 5) - 4(4 - 20) + 1(2 - 12)$$

$$= 3(1) - 4(-16) + 1(-10)$$

$$= 3 + 64 - 10 = 57$$

Notation. $$\sum_{j=1}^{n} A_{kj}\alpha_{ij}$$

MAIN THEOREM. The value of a determinant is equal to the sum of the products of the elements of any row by their corresponding cofactors.

The proof of this Main Theorem must be carried on by induction and is sufficiently difficult to be put off to another course, but the student is urged to write any third order determinant, and to evaluate it in a number of ways. Note that by a judicious application of the theorems above, the process of evaluating a determinant can be considerably shortened, by obtaining equivalent determinants with some zero elements.

Notation. From the Main Theorem:

$$\Delta = \sum_{i=1}^{n} A_{ij}\alpha_{ij} = \sum_{j=1}^{n} A_{ij}\alpha_{ij}$$

Example. We may evaluate the determinant of the example above by using the element of the second row:

$$-2\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} - 5\begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix} = -2(7) + 3(2) - 5(-13) = -14 + 6 + 65 = 57$$

or of the third column:

$$1\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = 1(-10) - 5(-13) + 2(1) = -10 + 65 + 2 = 57$$

Exercises. [I can supply as many as we think necessary.]

Chapter II

Supplement B

FLOW CHART FOR TWO LINEAR EQUATIONS IN X AND Y

Suppose we want to study the possible geometric relations between the graphs of two linear equations

$$L_1 : a_1x + b_1y + c_1 = 0$$
$$L_2 : a_2x + b_2y + c_2 = 0$$

Suppose further that we want the study to cover all pairs of ordered triples of real numbers (a_1,b_1,c_1) and (a_2,b_2,c_2). If we agree to include all such pairs, the study can easily be converted to a computer program and the coefficients themselves can even be generated internally in the computer as a part of a larger program.

If we know that the equations are not degenerate (i.e., either the x or y coefficient is different from zero), each represents a line in the plane, and these lines may be identical, parallel or intersecting. What we want to construct is an ordered set of questions we can ask about the coefficients of L_1 and L_2 which will distinguish for us how the graphs would have looked if we had drawn them. Our questions must be phrased in such a way that each answer will be either "yes" or "no."

Of course many different patterns of questions are possible. In general we want the pattern to branch like a tree with each question so that if an answer is "yes", the succeeding path will be different than it would have been had the answer been "no." At the end of each path will be a message stating the correct geometric configuration for the pair of equations with which we started. This type of pattern is often called a flow chart and is a useful tool in computer programming. If you think a little you will see that the well known game of Twenty Questions uses a kind of oral flow chart to solve the problem "What am I thinking of?"

Let us consider what the first question in our series should be. If at least one of the given equations is degenerate, then we do not really have two lines to study. We want to design our pattern to channel such equations

aside. Accordingly the first question might be

$$\boxed{\text{Is } (|a_1| + |b_1|) \cdot (|a_2| + |b_2|) = 0 \text{ ?}}$$

If the answer is "yes", then we know that either $|a_1| + |b_1| = 0$ or $|a_2| + |b_2| = 0$. In other words at least one equation is not really linear. We place the message, "Degenerate equation" and end this path. If the answer to the question was "no", we are assured of two linear equations. What shall we ask next? A possible second question is

$$\boxed{\text{Is } a_1b_2 - b_1a_2 \neq 0 \text{ ?}}$$

Notice that this time we ask whether a certain expression is different from zero. Of the answer is "yes", then we know the lines L_1 and L_2 intersect in a point. We write a message to this effect and close the path. If the answer to the second question is "no", then the two lines must be either parallel or coincident. We need a third question which will distinguish between these two cases. One such question is

$$\boxed{\text{Is } |a_1c_2 - a_2c_1| + |c_1b_2 - c_2b_2 - c_2b_1| = 0 \text{ ?}}$$

An answer of "yes" guarantees that $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = 0$ and $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 0$.

Therefore we have a pair of coincident lines. An answer of "no" in a similar way insures that L_1 and L_2 are parallel.

Let us repeat these three questions together with the message pattern we have indicated.

FLOW CHART

$$L_1 : a_1x + b_1y + c_1 = 0$$

$$L_2 : a_2x + b_2y + c_2 = 0$$

Is $(|a_1| + |b_1|) \cdot (|a_2| + |b_2|) = 0$? — yes → Either L_1 or L_2 is degenerate

↓ No

Is $a_1b_2 - b_1a_2 \neq 0$? — yes → L_1 and L_2 are intersecting lines

↓ No

Is $|a_1c_2 - a_2c_1| + |c_1b_2 - c_2b_1| = 0$? — yes → L_1 and L_2 are coincident lines

↓ No

L_1 and L_2 are parallel lines.

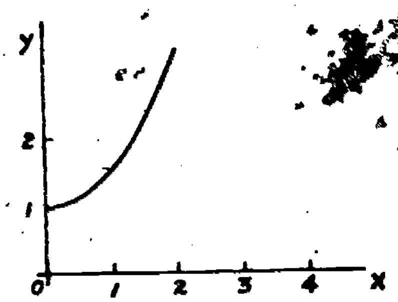
Chapter III

Supplement C

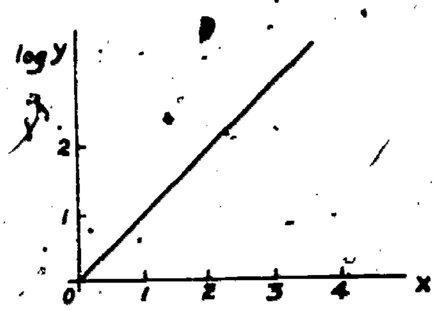
GRAPHS WITH NON-UNIFORM SCALES

In practice, it is sometimes necessary to graph a function $f(x,y)$ in a system of reference in which the axes are perpendicular to each other, but a different unit is used on each axis. For example, if the range of a function is <u>very</u> large compared to the domain, any unit small enough to allow the range to be graphed on a piece of paper will compress the domain too much to be helpful. We can study many properties of such a graph, but we must be careful never to read slopes from it without taking the difference of scale into account.

Other interesting variations of graphing $f(x,y)$ using perpendicular axes are semi-logarithmic and logarithmic graphs which prove to be helpful in applications of mathematics to biology, economics, and other sciences, especially where growth is involved. As an example, let us look at the graph of $y = e^x$ first in regular rectangular and then in semi-logarithmic coordinates.



(a)



(b)

Graph (a) is the familiar exponential function studied in Intermediate Mathematics. If $y = e^x$, then x is the natural logarithm of y or $x = \log y$. Clearly there is a linear relation, not between x and y, but between x and $\log y$. If we treat x as usual, and graph not y but $\log y$ on the vertical axis, we do indeed have a straight line. (See graph b.) This is called a semi-logarithmic graph because one of the axes measures the logarithm of a variable, rather than the variable itself.

If we go one step further and plot the logarithm of x on one axis and the logarithm of y (to the same base) on the other axis, we have a logarithmic graph. This type is used extensively in finding equations to fit experimental data when there is reason to believe the relationship is of the form $y = x^k$. Taking the logarithm of each side we have

$$\log y = k \log x .$$

If we graph our exponential data by measuring $\log y$ on one scale and $\log x$ on the other, we should be able to fit a straight line to the data, and determine k as the slope of the line.

As a matter of fact, if a scientist suspects his data could be described by either $y = a^x$ or $y = x^a$, he can plot the data using semi-logarithmic and full logarithmic coordinates. If either graph appears to be a straight line, his problem is solved. If the semi-logarithmic graph is a straight line, then $\log y = (\log a)x$, the slope is the logarithm of the base a, and the data is related by $y = a^x$. If the double logarithmic scale yields a straight line, then the slope, a, determines the exponent in the equation $y = x^a$ which relates the data.

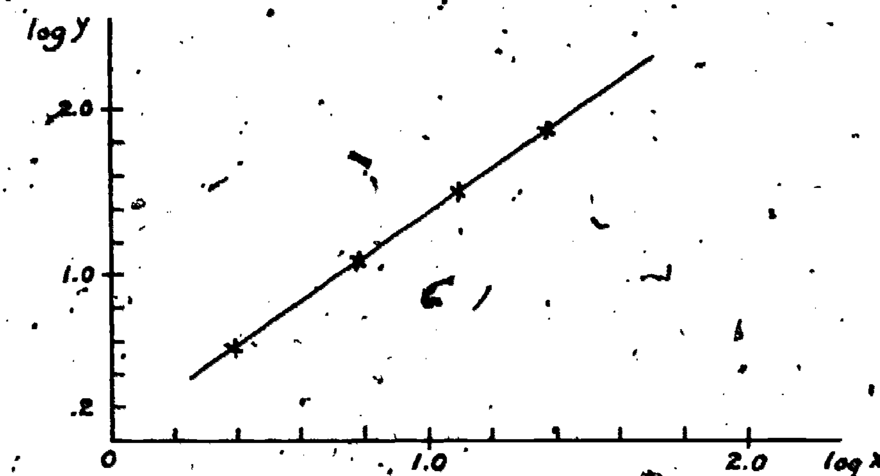
Problem. Suppose you have experimentally determined the following data and want to discover the mathematical relation between x and y.

x	2.50	6.20	11.6	21.4
y	3.61	12.9	30.9	72.9

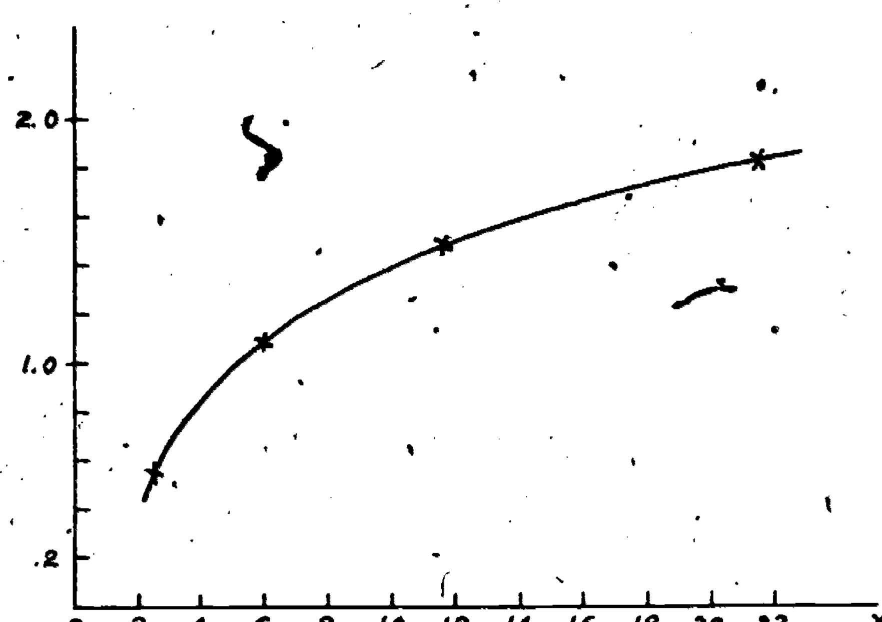
Suppose, further, you guess that y is either an exponential function involving x or that it is a power function of x.

Solution. Using common logarithms we fill out a table and plot the ordered pairs $(x, \log y)$ on one graph and $(\log x, \log y)$ on a second. Then we study the points and if either graph is approximately a straight line, we measure its slope. Finally we use this to express the relation between x and y.

x	2.50	6.20	11.6	21.4
log x	.398	.792	1.06	1.33
y	3.61	12.9	30.9	72.9
log y	.557	1.09	1.49	1.86



GRAPH II



The first graph seems to be linear and its slope $\frac{\Delta y}{\Delta x}$ is approximately $\frac{1.31}{.94} \sim 1.4$. Therefore, $\log y = 1.4 \log x$ or $y = x^{1.4}$ is the relation we are seeking.

Supplement to Chapter 2

COORDINATES AND THE LINE

S2-1.

From the postulates of geometry we deduced immediately that any point on a line may be chosen as the origin for a coordinate system and that the positive coordinates may be assigned to the interior points of either ray determined by the origin. However, in our development of the SMSG Geometry there need be no mention of units, in terms of which these measurements are made; the entire development depends upon one intrinsic scale of measure. For this reason we shall describe such coordinate systems as <u>intrinsic coordinate systems</u>. It would be very convenient to be free to choose coordinate systems with different scales of measure. It is easy to show that we have this freedom.

The coordinate system is an unusual type of function whose domain is the set of points on the line and whose range is the set of real numbers. Let us denote this function by f, whose value at each point X is the number $f(X) = x$. Let us consider a linear function, g, on the real numbers, defined by the equation $x' = g(x) = ax + b$, where a is any non zero real number and b is any real number. The composite function which assigns to each point X the number $g\big(f(X)\big)$ is also a one-to-one correspondence between the points of the line and the real numbers. We shall describe such correspondences as <u>linear coordinate systems</u>. We shall continue to describe the number which corresponds to a point as the <u>coordinate</u> of the point, since this phrase has meaning only with reference to a particular coordinate system. We shall denote the composite function of f by g as $g(f)$.

We shall consider the description of the geometric properties of the line in terms of such a linear coordinate system. Is there anything in a linear coordinate system comparable to the measure of distance between two points, R and S, whose coordinates in an intrinsic coordinate system on the line $\overleftrightarrow{RS}$ are r and s respectively? The new coordinates r' and s', of R and S respectively, are related by the equations

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$$r' = ar + b$$
$$s' = as + b .$$

We discover that

$$\begin{aligned} |r' - s'| &= |(ar + b) - (as + b)| \\ &= |ar - as| \\ &= |a| \cdot |r - s| . \end{aligned}$$

Unless $|a| = 1$, $|r' - s'|$ is not equal to $|r - s|$, the measure of distance in the intrinsic coordinate system. However, we do note that in the linear coordinate system, related to the intrinsic coordinate system by the equation $x' = ax + b$, the number $|r' - s'|$ is a constant multiple of $|r - s|$, the constant being independent of the choice of points.

We recall that the length of a segment was defined to be the measure of distance between its endpoints and that congruent segments were defined as segments having the same length. Thus the statement $\overline{RS} \cong \overline{TU}$ is equivalent to the statement, $|r - s| = |t - u|$, where r , s , t , and u are intrinsic coordinates of R , S , T, and U respectively.

If $$|r - s| = |t - u| ,$$

then $$|a| \cdot |r - s| = |a| \cdot |t - u| ,$$

$$|ar - as| = |at - au| ,$$

and $$|(ar + b) - (as + b)| = |(at + b) - (au + b)| ,$$

or $$|r' - s'| = |t' - u'| ,$$ where r' , s' , t' , and u' are coordinates in any linear coordinate system. Thus the condition defining congruence for segments applies in any linear coordinate system.

The student should think through all the details of the argument that any linear coordinate system is a one-to-one correspondence between the points of the line and the real numbers. Let f be an intrinsic coordinate system on a line L and let X by any point of L . Then $f(X)$ is a unique real number and so is $g(f(X)) = af(X) + b$. So far we have not used the assumption that $a \neq 0$. Now let r be a real number. Since $a \neq 0$, there is a unique number x_0 such that $ax_0 + b = r$. Since the original coordinate system is a one-to-one correspondence between the points of L and the real numbers, there is a unique point X_0 such that $f(X_0) = x_0$. Hence there is a unique point X_0 on L such that $g(f(X_0)) = g(x_0) = ax_0 + b = r$.

Example. Let P, Q, R, and S be four points on a line with intrinsic coordinates 2, 5, 8, and 11 respectively. Since $|2 - 5| = |8 - 11|$, $\overline{PQ} \cong \overline{RS}$. Let a linear coordinate system be defined by the equation $x' = 2x - 1$. Then the new coordinates of P, Q, R, and S are 3, 9, 15, and 21 respectively. Since $|3 - 9| = |15 - 21|$, the congruence of $\overline{PQ}$ and $\overline{RS}$ is similarly described in terms of the new coordinates.

The other geometric property described in terms of intrinsic coordinate systems on a line is betweenness on the line. We recall that the point S is between R and T if and only if $r < s < t$ or $r > s > t$, where r, s, and t are the coordinates of R, S, and T respectively. We observe that

if $$r < s < t$$

then $ar < as < at$ if $a > 0$, or $ar > as > at$ [illegible] < 0

and $$ar + b < as + b < at + b \quad \text{if} \quad a > 0$$

or $$ar + b > as + b > at + b \quad \text{if} \quad a < 0 .$$

The members of these inequalities are precisely the coordinates r', s', and t', which would be assigned to the points R, S, and T by a linear coordinate system defined by a linear equation $x' = ax + b$. Thus the last two lines of the above development may be replaced by

$$r' < s' < t' \quad \text{if} \quad a > 0 \text{, or } r' > s' > t' \quad \text{if} \quad a < 0 .$$

A similar argument obtains if $r > s > t$. In all cases the condition describing betweenness on a line holds if r, s, and t are replaced by the corresponding coordinates in any linear coordinate system.

The geometric properties of congruence for segments and betweenness on a line are described in exactly the same way in terms of linear coordinate systems as in the intrinsic coordinate systems. We summarize these results from the preceding two paragraphs as follows.

Any intrinsic coordinate system will not be changed under composition with the trivial linear function defined by the equation $x' = x$, and consequently is included among the linear coordinate systems on the line. These are the coordinate systems which are of use and interest to us. Henceforth, we shall usually consider only linear coordinate systems; where there is no chance of ambiguity we shall call these systems <u>coordinate systems</u>.

THEOREM S2-1. If a coordinate system on a line assigns the coordinates r, s, and t to the points R, S, and T, then S is between R and T if and only if $r < s < t$ or $r > s > t$.

THEOREM S2-2. Let P and Q be any two distinct points on a line. In a coordinate system C on the line, the coordinates of P and Q are p and q respectively. Let r and s be any two distinct real numbers. Then there exists a coordinate system C' on the line in which the coordinates of P and Q are r and s respectively.

Proof. We wish to discover whether there exists a linear function which relates C' to C by composition. If there is such a function, there exists an equation $x' = ax + b$ defining the function. The following equations would have to be satisfied.

(1) $$r = ap + b$$

and $$s = aq + b .$$

Combining equations, we obtain

$$r - s = a(p - q)$$

or $$a = \frac{r - s}{p - q} .$$

Substituting in Equation (1), we obtain

$$r = \frac{r - s}{p - q} \cdot p + b$$

or $$b = r - \frac{pr - ps}{p - q}$$

$$= \frac{pr - qr - pr + ps}{p - q}$$

$$= \frac{ps - qr}{p - q} .$$

The solution set for a and b of this pair of equations is $\{(\frac{r - s}{p - q}, \frac{ps - qr}{p - q})\}$. The coordinate system C' formed by the composition of C by the linear function defined by

$$x' = (\frac{r - s}{p - q})x + \frac{ps - qr}{p - q}$$

does satisfy the conclusion of the theorem. Since $p \neq q$, this equation, and consequently the coordinate system C', is always defined. In C' the coordinates of P and Q are given respectively by

$$p' = \left(\frac{r - s}{p - q}\right)p + \frac{ps - qr}{p - q} = \frac{pr - qr}{p - q} = r$$

and

$$q' = \left(\frac{r - s}{p - q}\right)q + \frac{ps - qr}{p - q} = \frac{ps - qs}{p - q} = s .$$

In fact, the coordinate system C' is unique, though we have not proved it here.

Corollary S2-2-1. If P and Q are any two distinct points on a line with coordinates p and q respectively in a coordinate system C, then the coordinate system C' which is related to C by the linear equation,

$$x' = \frac{1}{q - p} \cdot x - \frac{p}{q - p} ,$$

assigns the coordinates 0 and 1 to the points P and Q respectively. It is sometimes convenient in later computations to write this result in the form $x' = \dfrac{x - p}{q - p}$.

In order to make intuitively more clear the role played by the constants a and b in the introduction of a new coordinate system, we consider what new coordinates are assigned to the origin and to the unit-point under composition by the linear function defined by the equation $x' = ax + b$.

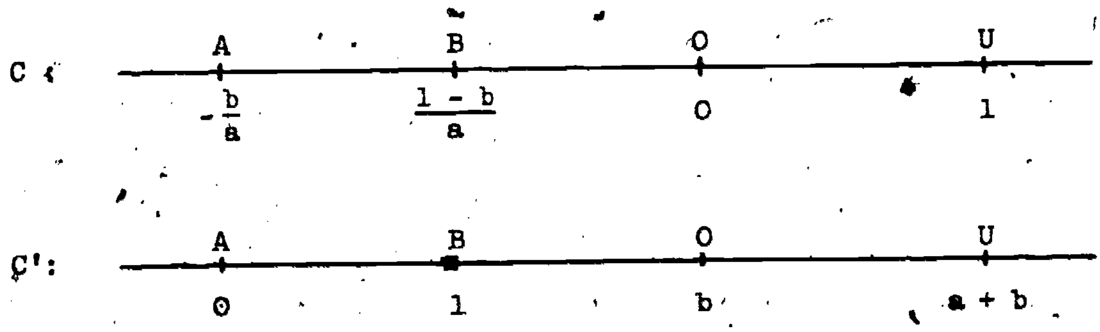


Figure S2-1

The point which was the origin now has coordinate b, and $d(O,U)$, which was 1 is now $|a|$. Thus the role of b is to shift the origin, and one role of a may be to increase or decrease the scale of distance. If $|a| > 1$, we say the new system is <u>scale-decreasing</u>; if $|a| < 1$, the new system is <u>scale-increasing</u>; if $|a| = 1$, the new system is <u>scale-preserving</u>. We observe that if $a > 0$ and the original coordinates p and q of two distinct points are unequal in the order $p < q$, then the new coordinates p' and q' are unequal in the order $p' < q'$, while if $a < 0$ and $p < q$, then p' and q' are unequal in the order $q' < p'$. For these reasons we say that the new system is <u>order-preserving</u> if $a > 0$ and <u>order-reversing</u> if $a < 0$.

Exercises S2-1a

Let P, Q, and R be points on a line with coordinates -5, 3, and 7 respectively. In Problems 1 - 6 find the coordinates of these points in the system given by composition of the original system by the linear function defined by the given equation. Is the new system scale-increasing, scale-decreasing, or scale-preserving? Is it order-preserving or order-reversing?

1. $x' = -x + 3$
2. $x' = 4x - 2$
3. $x' = \frac{1}{4}x + \frac{1}{4}$
4. $x' = -3x$
5. $x' = -\frac{2}{3}x + \frac{7}{3}$
6. $x' = x + 7$
7. For the systems described in Problems 1 - 6, find the coordinates of the points which were the origins and unit-points in the original system.
8. Find the original coordinates of the points which become the origin and unit-point of the systems described in Problems 1 - 6.
9. The equation $x' = ax + b$ defining the linear function which relates coordinate systems was subject to the condition $a \neq 0$. Why?

We have not considered the case in which we employ a non-linear equation to define a new coordinate system on a line, but it is interesting to do so. In Problems 10 - 13 the rules defining several functions of other types are given. Examine the coordinate system obtained by the composition of an intrinsic coordinate system and the function defined by the given equation. Does the coordinate system still describe betweenness on the line? Does it describe the congruent segments of the line adequately?

10. $x' = ax^3 + b$

11. $x' = e^x$

12. $\begin{cases} x' = \frac{1}{x} & \text{where } x \neq 0 \\ x' = x & \text{where } x = 0 \end{cases}$

13. $x' = \log_{10} x$

An important mathematical structure which you may have encountered only briefly is the group. A <u>group</u> is a set of elements with a binary operation which has the following properties:

Let S denote the set, a, b, and c, any elements of S, and $\circ$ the binary operation.

(1) (Closure) $a \circ b$ is a unique element of S

(2) (Associativity) $(a \circ b) \circ c = a \circ (b \circ c)$

(3) (Identity) S contains an element e such that $a \circ e = e \circ a = a$

(4) (Inverse) For each a there exists a' such that $a \circ a' = a' \circ a = e$.

An element e described in (3) is called an <u>identity</u> and an element a' described in (4) is called an <u>inverse</u> of a.

Some familiar examples of groups are the integers, the rational numbers, or the real numbers with addition as the operation. Other examples are the non-zero rational numbers or non-zero real numbers with multiplication as the operation.

Let us consider the set whose elements are the functions whose domains are the set of real numbers and which are defined by the equations $f(x) = ax + b$ where a is any non-zero real number and b is any real number. This set of functions forms a group under the binary operation of composition.

We shall prove that the identity and inverse properties are satisfied, but we leave the discussion of the closure and associative properties as exercises.

If the set contains an identity, it must be a function defined by a linear equation $g(x) = sx + t$. If this function is an identity, it must satisfy the following equation:

$$f(g(x)) = g(f(x)) = f(x) .$$

This becomes $\quad a(sx + t) + b = s(ax + b) + t = ax + b$

or $\quad asx + at + b = sax + sb + t = ax + b .$

This will be true if

(1) $\quad asx = sax = ax$, and

(2) $\quad at + b = sb + t = b .$

Since $a \neq 0$, Equation (1) will be true only if $s = 1$. Equation (2) thus becomes

$$at + b = b + t = b .$$

This equality implies that $t = 0$. Thus, the desired function $g(x) = sx + t = x$. There is only one function of this form. It is in the set, and it can be seen that it is an identity.

Now we want to find inverses. If an element, $f(x) = ax + b$, of the set has an inverse, it must be a function defined by a linear equation $g(x) = sx + t$. If this function is the inverse of $f(x)$, it must satisfy

$$f(g(x)) = g(f(x)) = x .$$

This becomes $\quad a(sx + t) + b = s(ax + b) + t = x$

or $\quad asx + at + b = sax + sb + t = x .$

This will be true if

(3) $\quad asx = sax = x$, and

(4) $\quad at + b = sb + t = 0 .$

Since $a \neq 0$, Equation (3) will be true if $s = \frac{1}{a}$. Equation (4) becomes

$$at + b = (\tfrac{1}{a}) \cdot b + t = 0 ,$$

which is true if $t = \frac{-b}{a}$, which is defined since $a \neq 0$.

Thus the desired function $g(x) = sx + t = (\frac{1}{a})x - \frac{b}{a}$. There is only one function of this form. It is in the set, and it can readily be shown to be an inverse of $f(x)$. In [illegible], identities and inverses are always unique, but we leave these questions as exercises.

Exercises S2-1b

1. Show that the set and binary operation described above have the closure property.

2. Show that the set and binary operation described above have the associative property.

3. Show that the set and binary operation described above do not have the commutative property.

4. Show that in any group the identity is unique.

5. Show that in any group the inverse of any given element is unique.

6. Show that in any group the inverse of the identity is the identity.

7. Let $f(x) = ax + b$ and $g(x) = px + q$. We denote the inverse of $f(x)$ by $f^{-1}(x)$. Find

   (a) $f\big(f(x)\big)$

   (b) $f\big(g(x)\big)$

   (c) $g\big(f(x)\big)$

   (d) $g\big(g(x)\big)$

   (e) $f\Big(f\big(f(x)\big)\Big)$

   (f) $g\Big(g\big(g(x)\big)\Big)$

   (g) $g^{-1}(x)$

   (h) $f^{-1}\big(g^{-1}(x)\big)$

   (i) $g^{-1}\big(f^{-1}(x)\big)$

   (j) the inverse of $f\big(g(x)\big)$

   (k) $g\big(f^{-1}(x)\big)$

   (l) $f\big(g^{-1}(x)\big)$

*8. Find the function (or functions) $h(x)$ such that

$$h\big(h(x)\big) = f(x) = ax + b .$$

Discuss the possibility and number of solutions for $h(x)$.

S2-2. Mappings and Linear Transformations.

A function whose domain is a set $\underline{A}$ and whose range is a set B (which may be the same as A) is frequently called a mapping. An element of the range which corresponds to a given element of the domain is said to be the image of that element. An element of the domain which corresponds to, or is mapped onto, a given element of the range is called a pre-image of that element.

In describing a mapping the second set mentioned may not always be the range of the function, but it always contains the range. If it is the range, the mapping is said to be onto the second set. If the range of the function is a proper subset of the second set, the mapping is said to be into the second set. A mapping is also called a transformation, especially when it is a mapping from a set of geometric entities into a set of geometric entities. The set of images corresponding to the elements of a given set in the domain is called the image set; the set of pre-images corresponding to the elements of a given set in the range is called the pre-image set.

The mappings which we consider in this section are one-to-one transformations of a line onto itself. We consider this line to have a fixed coordinate system. We need such a coordinate system to describe the transformation. We shall consider four types of transformations; translations, reflections, expansions, and contractions.

Intuitively, we may think of a translation as a shifting of the line along itself. A reflection is a half-rotation of the line about the origin. Expansions and contractions are uniform stretching from and shrinking toward the origin. We may describe these more explicitly.

DEFINITIONS. Let ℓ be a line with a coordinate system; let P be a point on the line with coordinate p ; let the point P' with coordinate p' be the image of P under a transformation of the line ℓ onto itself.

A transformation $T(P) = P'$ is a translation if and only if there exists a real number b such that for every point P , $p' = p + b$.

A transformation $R(P) = P'$ is a reflection if and only if for every point P , $p' = -p$.

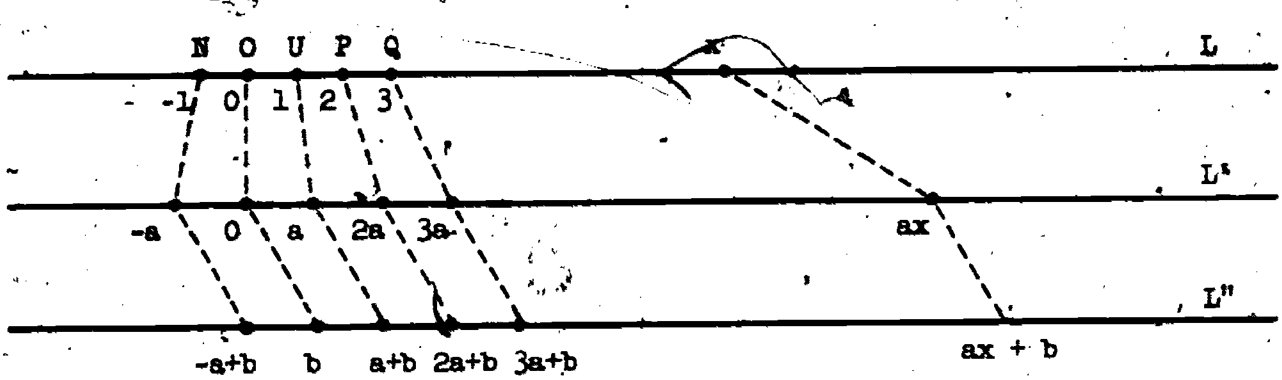
A transformation $E(P) = P'$ is an expansion if and only if there exists a real number $a > 1$ such that for every point P, $p' = ap$.

A transformation $C(P) = P'$ is a contraction if and only if there exists a positive real number $a < 1$ such that for every point P, $p' = ap$.

It should be intuitively apparent that in any of the above transformations an image is between two other images if and only if its pre-image is between the pre-images of the other two images. Therefore, the image set of a segment is also a segment. It should also be apparent that in a translation or a reflection, image segments are congruent if and only if the pre-image segments are congruent. It may or may not be clear that this is also the case in an expansion or contraction. We consider two congruent segments $\overline{PQ}$ and $\overline{RS}$. Their congruence depends upon the equality of $|p - q|$ and $|r - s|$. The congruence of the image segments depends upon the equality of $|p' - q'|$ and $|r' - s'|$. These may be expressed as $|ap - aq| = a|p - q|$ and $|ar - as| = a|r - s|$. These latter numbers are certainly equal if the original segments were congruent. Thus, the image segments of congruent segments are also congruent.

We continue our development by considering compositions of these transformations. A reflection maps a point X onto a point whose coordinate is $-x$; a translation will now map the new point onto a point whose coordinate is $-x + b$. An expansion maps a point X onto a point with coordinate ax; a translation now maps this new point onto a point whose coordinate is $ax + b$.

Such a sequence of transformations may be indicated in a diagram:



It should be understood that L , L' , and L'' are the same line, drawn in separate positions to show the transformations clearly. L' is the result of an <u>expansion</u> transformation of L , with the equation $x' = ax$, $(a > 1)$; L'' is the result of a <u>translation</u> transformation of L' , with the equation $x'' = x' + b$; finally, L'' can be considered as the result of a composition of two transformations of L , with the equation $x'' = ax + b$.

We consider the successive application or composition of two of these transformations and display the results by means of the table below. We employ the notation used in the definitions given above. The labels at the top indicate which transformation is performed first; the labels on the left indicate which transformation is performed second. The entry is the coordinate of the image of a point X , subject to the restrictions of the given transformations. The subscripts of the constants indicate which transformation introduced them.

	T	R	E $(a_1 > 1)$	C $(0 < a_1 < 1)$
T	$x + b_1 + b_2$	$-x + b_2$	$a_1x + b_2$	$a_1x + b_2$
R	$-x - b_1$	x	$-a_1x$	$-a_1x$
E $(a_2 > 1)$	$a_2x + a_2b_1$	$-a_2x$	a_1a_2x	a_1a_2x
C $(0 < a_2 < 1)$	$a_2x + a_2b_1$	$-a_2x$	a_1a_2x	a_1a_2x

We summarize by observing that these transformations and the transformations that may be obtained from them by composition may be included in the set of transformations defined as follows:

<u>DEFINITION</u>. Let ℓ be a line with a coordinate system; let P be a point on the line with coordinate p ; let the point P' with coordinate p' be the image of P under a transformation of the line ℓ onto itself.

A transformation $T(P) = P'$ is a <u>linear</u> <u>transformation</u> if and only if there exist a non-zero real number a and a real number b such that for every point P, $p' = ap + b$.

We call these mappings linear transformations because the defining equations are linear.

If this argument has not begun to sound familiar, you should go back to Section 2-1.

The set of linear transformations of a line onto itself under the binary operation of composition is another instance of a group.

Exercises S2-2a

In the following exercises, you may find that the <u>form</u> of the proofs you are asked to give are remarkably similar, if not identical, to those in Section 2-1. They are different only in interpretation and terminology.

1. Prove that if Q is between P and R, then in a linear transformation of PR onto itself, the image of Q is between the images of P and R.

2. Prove that if $\overline{PQ}$ and $\overline{RS}$ are congruent segments contained in a line, then in a linear transformation of the line onto itself $\overline{P'Q'} \cong \overline{R'S'}$, where P', Q', R', and S' are the images of P, Q, R, and S respectively.

3. Prove that the set of linear transformations of a line onto itself is closed under composition.

4. Prove that the operation of composition is associative for linear transformations of a line onto itself.

5. Prove that the set of linear transformations of a line onto itself contains an identity with respect to composition.

6. Prove that each element of the set of linear transformations of a line onto itself has an inverse with respect to composition.

7. Prove that the composition of linear transformations of a line onto itself is not commutative. The composition is commutative if certain restrictions are placed on the linear transformations. What are these restrictions?

8. Prove that any linear transformation may be expressed as the composite of not more than three transformations each of which is a translation, a reflection, a contraction, or an expansion.

Although there is no unique way of "factoring" a linear transformation in the way suggested above, it may be that for a given transformation every such expression must include a translation, a reflection, an expansion, or a contraction. In this case we shall say that the linear transformation includes a translation, reflection, or expansion.

We have discovered that the linear transformations of a line onto itself under the binary operation of composition form a group which seems similar to the group of linear functions which describe changes of coordinate system on a line under the binary operation of composition.

This kind of similarity is of some importance in mathematics and is called an isomorphism (from the Greek, ισος, meaning same, and μορφη, meaning form). An isomorphism is a one-to-one correspondence between two mathematical structures which relates not only the elements of the structures but also the operations between the elements. A familiar example is found in the relationship between the multiplication of positive real numbers and the addition of their logarithms. Another example is found in the relationship between the addition of vectors and the addition of complex numbers. The importance of isomorphisms stems from the fact that statements made about one structure may suggest corresponding statements about the other.

In this case the isomorphism is between the group of linear transformations of the line onto itself under composition and the group of changes of coordinate system on the line under composition. The correspondence is established by identical linear functions which occur in the definition of each group. Since our descriptions of each group are in terms of linear functions defined by equations of the form $x' = ax + b$, we may make comparisons of our descriptions when the conditions on a and b are the same.

A change of coordinate system which shifts the origin corresponds to a linear transformation which includes a translation. A change of coordinate system which is measure-preserving corresponds to a linear transformation which includes only a translation or a reflection. A change of coordinate system which is measure-increasing corresponds to a linear transformation which includes a contraction, and a change of coordinate system which is measure-decreasing corresponds to a linear transformation which includes an

expansion. A change of coordinate system which is order-preserving corresponds to a linear transformation which does not include a reflection, and a change of coordinate system which is order-reversing corresponds to a linear transformation which includes a reflection.

Lastly, we consider whether a point may be assigned the same coordinate after a change of coordinate system. The comparable situation for a transformation is that a point is mapped onto itself. In either case, where $x' = ax + b$, the situation occurs if $x' = x$.

If $$x' = x,$$

then $$x' = ax + b$$

becomes $$x = ax + b$$

or $$(a - 1)x = -b.$$

If $a = 1$ and $b = 0$, we have the identical coordinate system (or the identity transformation) in which all coordinates (or points) are unchanged; if $a = 1$ and $b \neq 0$, there is no coordinate (or point) which is unchanged. If $a \neq 1$, the coordinate (or point with coordinate) $\frac{-b}{a - 1}$ is unchanged. It is customary to say that such numbers or points are <u>fixed</u> or <u>invariant</u>.

Exercises S2-2b

1. Prove that a change of coordinate system is order-preserving if and only if $\frac{r' - s'}{r - s}$ is positive, where r' and s' are the new coordinates of points whose original coordinates were r and s respectively; prove that a change of coordinate system is order-reversing if and only if $\frac{r' - s'}{r - s}$ is negative.

2. Consider a linear transformation of a line onto itself which maps the points R and S, whose coordinates are r and s respectively, onto the points whose coordinates are r' and s' respectively. Prove that the transformation includes:

   (a) a contraction if and only if $0 < \frac{r' - s'}{r - s} < 1$.

   (b) a contraction and a reflection if and only if $-1 < \frac{r' - s'}{r - s} < 0$

(c) An expansion if and only if $\frac{r' - s'}{r - s} > 1$

(d) an expansion and a reflection if and only if $\frac{r' - s'}{r - s} < -1$.

3. Consider a linear transformation of a line onto itself which maps the points P and Q, whose coordinates are p and q respectively, onto the points whose coordinates are p' and q' respectively. Prove that the transformation includes:

(a) a translation if and only if $\frac{p' - q'}{p - q} = 1$

(b) a reflection if and only if $\frac{p' - q'}{p - q} = -1$.

4. Show that the intrinsic coordinate systems on a line are identical to the linear coordinate systems whose defining functions have the form $x' = x + b$ and $x' = -x + b$, where b is any real number.

5. Consider a line with a coordinate system, let P be a point of the line and let $I(P) = P'$ be the image of P under a transformation of the line onto itself; let p and p' be the coordinates of P and P' respectively.

Consider the transformation defined by

$I(P) = P'$ where $p' = \frac{1}{p}$ for $p \neq 0$, and $p' = p$ for $p = 0$.

Choose an appropriate scale and make a graph for the coordinate system; write the coordinates of several images below. Write the coordinates of their corresponding pre-images above them. A transformation of this type is called an <u>inversion</u> of the line.

6. Consider the composition $F(G(H))$ of transformations of a line to itself, where W , X , Y , Z are points of the line with coordinates w , x , y , and z respectively, and

$F(Y) = Z$ where $z = \frac{1}{y}$ for $y \neq 0$, and $z = y$ for $y = 0$,

$G(X) = Y$ where $y = x + 1$, and

$H(W) = X$ where $x = 2^{w}$

(a) Describe the set of pre-images, or domain, and the set of images, or range, of the composite transformation in terms of the coordinate system on the line. Is this transformation into or onto the line? Is this a one-to-one mapping?

(b) Choose an appropriate scale for the coordinate system and make a graph of the set of images of this composite transformation. Write the coordinates of several images below them. Write the coordinates of their corresponding pre-images above them.

(c) Two sets are said to have the same cardinal number or the same cardinality if their elements may be put in one-to-one correspondence. What can you say about the cardinality of the interior of a segment of a line?

7. Consider the composition $D(E(F))$ of the functions whose domains are the set of real numbers, where

$$z = D(y) = \begin{cases} \frac{1}{y} & \text{for } y \neq 0 \\ y & \text{for } y = 0 \end{cases}$$

$$y = E(x) = x + 1 \quad \text{for all } x$$

$$x = F(w) = 2^w \quad \text{for all } w .$$

(a) Describe the domain and range of the composite function. Is this mapping into or onto the set of real numbers? Is this mapping one-to-one?

(b) The cardinality of a set is said to be infinite if and only if the elements of the set may be put into one-to-one correspondence with the elements of a proper subset of the given set. What can you say about the cardinality of the set of real numbers?

8. If P, Q, R, and S are points, with $R \neq S$, whose respective coordinates in two different coordinate systems are p, q, r, s and p', q', r', s', prove that

$$\frac{p' - q'}{r' - s'} = \frac{p - q}{r - s} .$$

Each member of the equation is called a difference quotient, and in this case expresses the ratio of a pair of directed distances. The content of this theorem might be expressed in this way:

> Difference quotients of directed distances are invariant under a change of coordinate system.

Or this way:

> The ratio of directed distances depends upon the points involved, but not upon the coordinate system.

9. If A, B, and C have respective coordinates 3, 5, and 10 in one coordinate system, and 2, 3, and x in another coordinate system, find x. (In how many ways can you do this problem?)

10. If A, B, and X are distinct points with respective coordinates a, b, x, and a', b', x' in two different coordinate systems, express x' in terms of a, b, a', b', and x.

11. Show that if two points are fixed under a linear transformation, it must be the identity transformation.

Supplement to Chapter 3

LINEAR DEPENDENCE AND INDEPENDENCE

We have defined a zero vector, $\vec{0}$, and, for any number k and vector $\vec{X}$, the scalar product $k\vec{X}$. We may, in the same way, define a zero linear polynomial in one variable, $0 + 0x$; and, for any number k and linear polynomial in one variable, $a + bx$, the "scalar product" $k(a + bx) = ka + kbx$. We could, in the same way, define a zero n-tuple of numbers, and, for any number k and any n-tuple of numbers, the "scalar product", $k(a,b,\ldots,n) = (ka,kb,\ldots,kn)$.

We consider now a set $S = \{A,B,\ldots,K\}$, whose members may all be vectors, or linear polynomials in one variable, or ordered n-tuples of numbers, etc.... We may see that, with suitable definitions along the lines suggested above, members of S might all be linear expressions in two variables, or polynomials in x of degree not greater than 3, or any polynomials in x, and so on.

A set of such expressions $S = \{A,B,\ldots,K\}$ is said to be linearly dependent (L.D.) if there exists a set of numbers $N = \{a,b,\ldots,k\}$, not all zero, such that $aA + bB + \ldots + kK = 0$.

Example. The set $\{2p + 3q, 6p + 9q\}$ is L.D. because there is a set of numbers $\{-3,1\}$ not all zero, such that $-3(2p + 3q) + 1(6p + 9q) = 0$.

If a set of expressions is not linearly dependent, it is said to be linearly independent (L.I.).

Example. The set $\{2p + 3q, 6p + 10q\}$ is L.I. because, if there were a set of numbers (a,b) such that $a(2p + 3q) + b(6p + 10q) = 0$, then we would have

$$(2a + 6b)p + (3a + 10b)q = 0$$

for all p and q, or

$$2a + 6b = 0 \text{ and } 3a + 10b = 0 .$$

The only solutions for these equations are $a = 0$, $b = 0$; therefore, the original set is not L.D., it is L.I.

In view of the example above, it is possible to define linear independence first, as some authors do.

A set of such expressions as $S = \{A,B,\ldots,K\}$ is said to be linearly independent (L.I.) if, for the set of numbers $N = \{a,b,\ldots,k\}$, the statement $aA + bB + \ldots + kK = 0$, implies $a = b = \ldots = k = 0$.

Terminology. The property of being L.D. or L.I. is a collective one, and attaches to the set, rather than to the separate individuals; however, we follow general usage in writing, sometimes, "The vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ are L.I." for the longer "The set of vectors $\{\vec{A},\vec{B},\vec{C}\}$ is L.I."

We state some useful theorems whose proofs are left to the reader.

THEOREM 1. A set is L.D. if any subset of it is L.D.

THEOREM 2. If a set with at least two members is L.D., then one member can be expressed as a linear combination of the others.

Corollary. If the set $\{A,B,\ldots,K\}$ is L.I., and the set $\{A,B,\ldots,K,L\}$ is L.D., then L can be expressed as a linear combination of A, B, $\ldots$, K.

THEOREM 3. If the set of rows (or columns) of a determinant is L.D., then the value of the determinant is zero.

Proof. If the set of rows is L.D., then one row, say, the first, may by Theorem 2 be expressed as a linear combination of the others.

The illustration below, with a determinant of order 3 is easily extended to any order.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{21} + \ell a_{31} & ka_{22} + \ell a_{32} & ka_{23} + \ell a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

But, by Theorem 5 of Supplement A, this last determinant may be written as a sum of determinants, and equals

$$\begin{vmatrix} ka_{21} & ka_{22} & ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} \ell a_{31} & \ell a_{32} & \ell a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} .$$

The application of Theorem 4 of Supplement A shows that both of these are equal to zero, and therefore, so is the original determinant.

<u>Application</u> <u>to</u> <u>vectors</u>.

<u>THEOREM</u> 4. Any set of vectors which includes the zero vector is L.D.

<u>THEOREM</u> 5. Two non-zero vectors are L.D. if and only if they are collinear.

(a) If $\vec{P}$ and $\vec{Q}$ are collinear, then, from Chapter 3, there exists a number k such that $\vec{P} = k\vec{Q}$. Therefore $1\vec{P} - k\vec{Q} = \vec{O}$, therefore $\vec{P}$ and $\vec{Q}$ are L.D.

(b) If $\vec{P}$ and $\vec{Q}$ are L.D. then there exist numbers a and b not both $= \vec{O}$, such that $a\vec{P} + b\vec{Q} = \vec{O}$. Suppose $a \neq 0$, then $\vec{P} = -\frac{b}{a}\vec{Q}$, that is $\vec{P} = k\vec{Q}$ which means that $\vec{P}$ and $\vec{Q}$ are collinear.

<u>Corollary</u>. $[p,q]$, $[r,s]$ are collinear if and only if $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 0$.

__THEOREM 6.__ In the plane, any set of three non-zero vectors is L.D.

(a) If any two are collinear, they are L.D. and then so is the set of three.

(b) If no two are collinear, then, for any c, we will show that we can always find values for a and b such that

$$a\vec{P} + b\vec{Q} + c\vec{R} = \vec{0} ;$$

that is we can find a, b, for any p, q, r, s, t, u, c such that

$$a[p,q] + b[r,s] + c[t,u] = [0,0] .$$

This requires unique solutions for a, and b, in the equations

$$pa + rb = -ct$$
$$qa + sb = -cu .$$

But, from the hypothesis that $\vec{P}$ and $\vec{Q}$ are not collinear, we have $\begin{vmatrix} p & r \\ q & s \end{vmatrix} \neq 0$, and this is exactly the condition that there be unique solutions for a and b in the equations above.

__Corollary.__ In the plane, __any__ vector can be expressed as a linear combination of any pair of non-collinear vectors. That is, if $\vec{P}$ and $\vec{Q}$ are not collinear, then, for any $\vec{X}$ we can find numbers a and b so that $a\vec{P} + b\vec{Q} = \vec{X}$. (Compare with Theorem 3-5).

__Terminology.__ If __any__ vector of the plane can be expressed as a linear combination of the members of some set $S = \{\vec{P},\vec{Q},\vec{R},\ldots\}$, then S is said to __span__ the plane. A set of vectors which is L.D. and which spans the plane is called a __basis set__, or simply a __basis__ for the plane.

Note: (1) Any pair of non-collinear vectors forms a basis for the plane.

(2) These concepts generalize in a natural and interesting way to higher dimensions;

The set of vectors, $\{[1,0], [0,1]\}$ is what is called the "natural basis" for the plane, since, $[a,b] = a[1,0] + b[0,1]$.

The natural basis for three dimensions is the set $\{[1,0,0], [0,1,0], [0,0,1]\}$; etc.

(3) The number of vectors in the basis is the same as the dimension of the space. Thus, we may define a space of four dimensions as one in which there is at least one set of four L.I. vectors, but in which every set of five vectors is L.D. Similar definitions may be stated for five and higher dimensions.

Applications to Geometry

1. The lines $ax + by = c$, and $px + qy = r$ intersect in a point if and only if the corresponding equations have a unique solution for x and y , that is, if and only if $\begin{vmatrix} a & b \\ p & q \end{vmatrix} \neq 0$. This is true if and only if the left members of these equations are L.I. If the left members are L.D. then the lines will be parallel or coincident, as can easily be shown.

2. The concept introduced above generalizes easily. The planes:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

meet in a single point if and only if the left members of these equations are L.I. If they are L.D. then the planes may be related in various ways. All three may be parallel, two or three of them may coincide, two may be parallel and intersect the third, they may intersect in three parallel lines, etc. We leave the interested student to discover, either by his own research or by reference to other books, the connection between the dispositions of the planes, and the relations among the coefficients in their equations.

Supplement D

Supplements to Chapters 2, 3, and 8

POINTS, LINES, AND PLANES

The material in this supplement previously appeared as Chapter 4 in the preliminary edition. Parts of that chapter were retained in the text you are now using. These sections include significant material which may be of interest to you.

D-1. Choice of Methods

In this chapter we shall consider some questions about the undefined elements of geometry -- points, lines, and planes. When do they intersect? How are they separated? What about betweenness? For answering these and other questions, we have developed the basic tools in the earlier chapters; it will be part of our task to select from among these tools those appropriate to the solution of a particular problem.

Sometimes we shall start with the general case and then take special cases. You may recall proving Desargues' Theorem in 3-space, and then showing that it holds in 2-space. At other times, we start with a more limited case and then generalize. Thus we considered distance first on a line, then in 2-space, and so on.

We have available different forms of representation. In a problem about a particular line, our representation of it may depend on what is known about it, what we want to prove about it, or other considerations. For example, if you are told that the x-intercept for a certain line is 2 and the y-intercept -3, you might choose as its equation, $\frac{x}{2} + \frac{y}{-3} = 1$. If you are concerned with the amount of rotation of a line about a fixed point, you might want to use that point as pole of a polar coordinate system and write for the line $\theta \doteq k$. A relation such as $r = \theta$, expressible most simply in polar coordinates,

would be much more complicated to look at and to graph in rectangular coordinates. (You might want to try this.) In Chapter 4 vector methods are used to prove theorems of geometry that you proved earlier in other ways.

Our point here is that in this text from this point on you can expect to see a variety of representations and methods. In Sections D-2 and D-3, for example, rectangular coordinates and the equation $ax + by + c = 0$ for a line are chosen because it is desired to emphasize the relation of the geometric problem to an algebraic problem of solving systems of equations. In the same fashion, you have freedom to select the form of representation and the method that seems appropriate in a particular problem. Sometimes a few minutes spent first in deciding how to locate a coordinate system will save much time in solving a problem. Often there is no single simplest or best method.

D-2. Collinearity.

The geometric problem of whether three points are collinear corresponds to the algebraic problem of whether three pairs of values of two variables are solutions of the same linear equation in two variables.

Consider distinct points $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$, $P_3 = (x_3,y_3)$. Using the two-point form of the equation of a line derived in Section 2-5, the equation of the line $\overleftrightarrow{P_2P_3}$ can be written

$$y - y_3 = \frac{y_2 - y_3}{x_2 - x_3}(x - x_3) .$$

This we rewrite as

$$(y - y_3)(x_2 - x_3) = (y_2 - y_3)(x - x_3) .$$

If we multiply out and collect terms involving x and y, we have

(1) $$(y_2 - y_3)x - (x_2 - x_3)y + (x_2y_3 - x_3y_2) = 0 .$$

If we write the terms in parentheses as second order determinants (Appendix A), (1) becomes

$$x\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - y\begin{vmatrix} x_2 & 1 \\ x_3 & 1 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = 0 .$$

Using x, y, and 1 as the elements of the first row of a third order determinant, we can then write the equation in the form

$$(2) \qquad \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 .$$

Since (2) is an equation of the line P_2P_3, the point P_1 is on this line if and only if

$$(3) \qquad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 .$$

Thus (3) is a compact form in which to write the condition that three points are collinear.

If three given points are not collinear, they determine a triangle. We choose a rectangular coordinate system so that the triangle is entirely in the first quadrant and name the points P_1, P_2, P_3, in a counterclockwise order around the triangle, as shown in Figure D-1.

If the points P_1, P_2, P_3 are not collinear, they determine a triangle. To find its area we draw perpendiculars P_1F_1, P_2F_2, P_3F_3 to the x-axis. We can find the area K of $\Delta P_1P_2P_3$ by subtracting the area of trapezoid $F_1P_1P_3F_3$ from the sum of the areas of trapezoids $F_1P_1P_2F_2$ and $F_2P_2P_3F_3$.

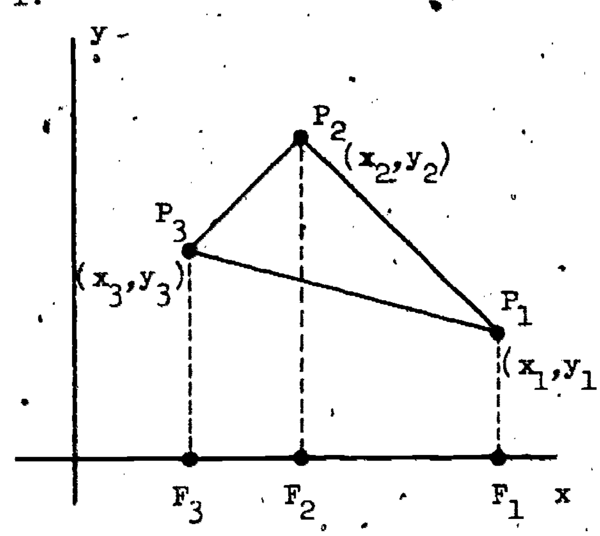


Figure D-1

$$K = \text{Area } F_1P_1P_2F_2 + \text{Area } F_2P_2P_3F_3 - \text{Area } F_1P_1P_3F_3 .$$

$$K = \tfrac{1}{2}(x_1 - x_2)(y_1 + y_2) + \tfrac{1}{2}(x_2 - x_3)(y_2 + y_3) - \tfrac{1}{2}(x_1 - x_3)(y_1 + y_3) ,$$

$$= \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1) ,$$

$$(4) \qquad K = \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] ,$$

$$(5) \text{ or } K = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| .$$

The student should verify that Equations (4) and (5) are equivalent. The value of the determinant in (5) will be positive if the vertices are named as in Figure D-1 so that traverse of the perimeter in the order $P_1P_2P_3$ is counterclockwise. If it is clockwise, the value of the determinant will be negative.

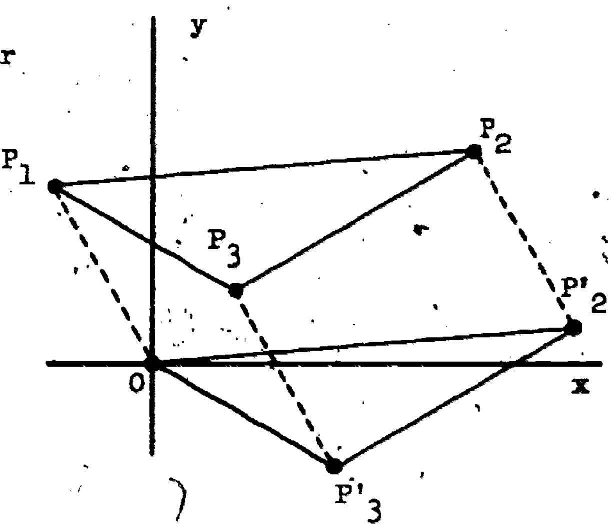
We notice that the determinant in (5) is the same as the one used to write (3), the condition that three points are collinear. This is not surprising, as it is intuitively obvious that three points are collinear if and only if the area of the "triangle" they determine is zero.

Formula (3) can be obtained in a different way by using vectors. In Section 3-8 we saw that the area of the triangle OXY, where $X = (x_1,x_2)$ and $Y = (y_1,y_2)$, is

$$K = \frac{1}{2}|x_1y_2 - x_2y_1| .$$

We use this result to find the area of an arbitrary triangle.

We name the vertices $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$, $P_3 = (x_3,y_3)$, so that our results shall have the same notation as the preceding development. We add the vector $-\vec{P}_1$ to each of the vectors $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$ to obtain the vectors $\vec{P}_1 - \vec{P}_1 = \vec{0}$, $\vec{P}_2 - \vec{P}_1 = \vec{P}_2'$, $\vec{P}_3 - \vec{P}_1 = \vec{P}_3'$ where



$$\vec{P}_2' = [x_2 - x_1 , y_2 - y_1]$$
$$\vec{P}_3' = [x_3 - x_1 , y_3 - y_1] .$$

Triangle $OP_2'P_3'$ is congruent to triangle $P_1P_2P_3$. Thus the area of triangle $P_1P_2P_3$ is

$$K = \frac{1}{2}\left|(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\right|$$

$$= \frac{1}{2}\left\|\begin{matrix} 0 & 0 & 1 \\ x_2 - x_1 & y_2 - y_1 & 1 \\ x_3 - x_1 & y_3 - y_1 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\| .$$

Exercises D-2

1. For each of the following find out whether the points whose coordinates are given are collinear; if not, find the area of the triangle that is determined.

   (a) $(7,0)$, $(4,-1)$, $(13,2)$ (c) (a,b) , $(-a,-b)$, (c,d)

   (b) $(3,2)$, $(-2,-7)$, $(15,5)$ (d) $(b,0)$, $(0,-b)$, $(a , a - b)$

2. Consider the triangle with vertices $P_1 = (0,0)$, $P_2 = (a,0)$, $P_3 = (b,c)$ and the value (not the absolute value) of the determinant in (5) . evaluate this determinant for P_1 , P_2 , P_3 . Evaluate it for $Q_1 = (0,0)$, $Q_2 = (b,c)$, $Q_3 = (a,0)$; for $R_1 = (a,0)$, $R_2 = (b,c)$, $R_3 = (0,0)$; and also for $S_1 = (b,c)$, $S_2 = (a,0)$, $S_3 = (0,0)$. Does the way you go around the triangle make a difference? Does the vertex at which you start make a difference? Try to state some general conclusions.

3. Prove that the area of the triangle with vertices $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$, $P_3 = (x_3,y_3)$ is

$$K = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right| .$$

Note: The equation above may be written

$$K = \frac{1}{2}\left|\sum_{i=1}^{3}\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1}\end{vmatrix}\right| ,$$

where we interpret x_4 as x_1 and y_4 as y_1.

This generalizes immediately, giving the following formula for the area of a polygon with n vertices $P_i = (x_i, y_i)$:

$$K = \frac{1}{2}\left|\sum_{i=1}^{n}\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1}\end{vmatrix}\right| ,$$

where we interpret x_{n+1} as x_1 and y_{n+1} as y_1.

4. Find the area of the quadrilateral whose vertices are $P_1 = (4,1)$, $P_2 = (-1,3)$, $P_3 = (-3,-28)$, $P_4 = (2,-1)$, first by adding the areas of $\Delta P_1P_2P_3$ and $\Delta P_3P_4P_1$, and then by using the formula in Problem 3 above.

5. Prove that points $A = (-2,1)$, $B = (2,-2)$, and $C = (6,-5)$ are collinear.

   (a) Use condition (3).

   (b) Show that $\bar{B} - \bar{A} = k(\bar{C} - \bar{A})$.

   (c) Show that $d(A,B) + d(B,C) = d(A,C)$.

D-3. Concurrence.

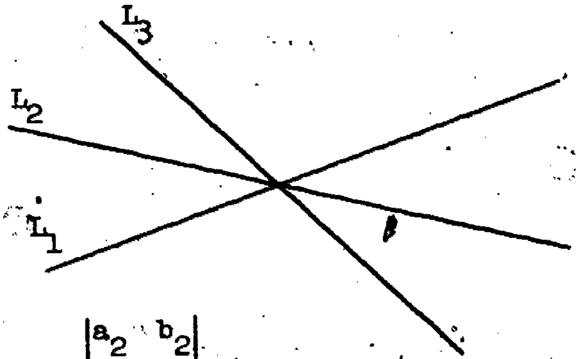
The geometric problem of whether three lines are concurrent corresponds to the algebraic problem of whether one pair of values of two variables satisfies three different linear equations in two variables.

We consider three lines L_1, L_2, and L_3, with equations

$$(1) \qquad \begin{aligned} L_1 &: a_1x + b_1y + c_1 = 0 \\ L_2 &: a_2x + b_2y + c_2 = 0 \\ L_3 &: a_3x + b_3y + c_3 = 0 . \end{aligned}$$

These lines may be related in any one of the following ways; we shall consider the analytic conditions for each.

(a) The lines may be concurrent. This is the case of most interest to us since it represents the usual situation in which there is a unique solution of the three equations. The equations represent three distinct lines with one and only one point in common. For this, any two of the lines must intersect in a point, and that point must lie on the third line. From our study of Intermediate Mathematics we know that this first requirement means that we must have



$$(2) \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 , \quad \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \neq 0 , \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \neq 0 .$$

The second condition requires that the intersection of, say, L_1 and L_2, must lie on L_3. If $P_1 = (x_1, y_1)$ represents the intersections of L_1 and L_2, we may write its coordinates

$$x_1 = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} , \quad y_1 = \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} .$$

The condition that P_1 is on L_3 is

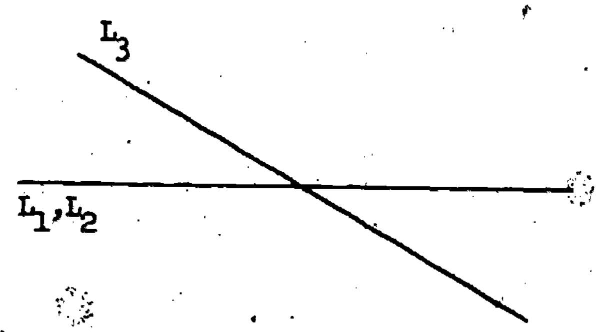
$$a_3 \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + b_3 \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + c_3 = 0 ,$$

which can be written more compactly as

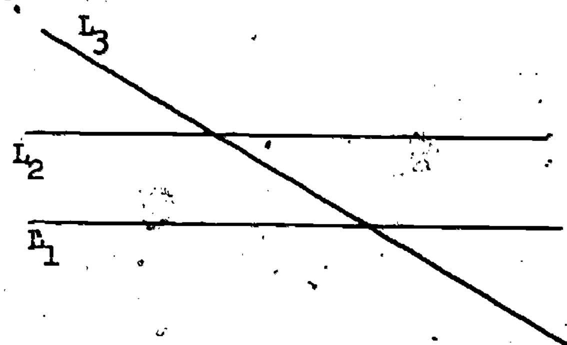
$$(3) \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 .$$

Thus the condition that three distinct lines be concurrent is that the determinant of their coefficients is zero.

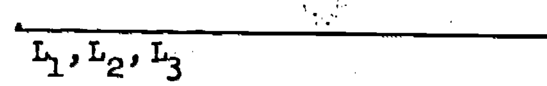
(b) Two lines may coincide and be intersected by the third line. In this case the third order determinant is zero and there is a unique solution of the three equations, but this case may be distinguished from (a) by noting that one of the determinants of (2) is zero.



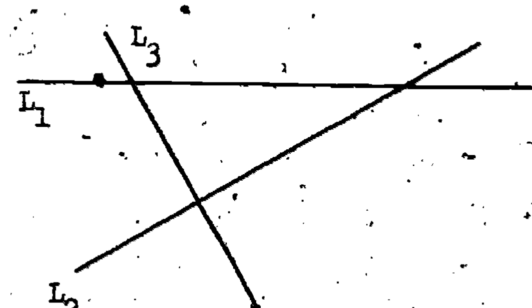
(c) Two lines may be parallel and be intersected by the third line. The student must be careful to distinguish this case from case (b), because here there is not a unique solution. This case resembles (b) in that one of the determinants of (2) is zero, but the determinant of the coefficients is not zero.



(d) The three lines may coincide. There is not a unique solution in this case since any solution of one equation is also a solution of each of the others. The third order determinant is zero as are all three determinants of (2). There are two other distinguishable cases which have these same algebraic conditions. The student may be interested in describing these cases and discovering how to distinguish them from case (d).



(e) Each line may intersect each of the others in a single point. Condition (2) holds, but the third order determinant is not zero. This is the case one is most likely to observe from three randomly chosen lines.



We might approach the question of concurrence in a somewhat different fashion. Let L_1 and L_2 be lines with equations given in (1). Then if m and n are any numbers not both equal to zero, the equation

$$(4) \qquad m(a_1x + b_1y + c_1) + n(a_2x + b_2y + c_2) = 0$$

is the equation of a line, since it is a first-degree equation in x and y.

If L_1 and L_2 intersect in $P_1 = (x_1,y_1)$, then (4) represents, for suitable choices of m and n, any line through P_1. If L_1 and L_2 are parallel, then (4) represents, for suitable choices of m and n, any line parallel to L_1 and L_2. If L_1 and L_2 coincide, then (4) represents that same line. Proof of these last statements will be left to the interested student.

Equation (4) represents what is often called a <u>family</u> of lines; that is, for suitable values of m and n it represents all the lines containing the intersection of L_1 and L_2. Thus a condition that three distinct lines (with equations in the form $ax + by + c = 0$) be concurrent is that the left member of the equation of one of them is a linear combination of the left members of the equations of the other two.

<u>Example 1</u>. Find a value of k for which lines with the following equations will be concurrent. (Assume $k \neq -1$)

$$x - y = 0$$
$$3x + 2 = 0$$
$$kx + y + 1 = 0.$$

<u>Solution</u>. We observe that the lines are not parallel (they satisfy condition (2)); we then use condition (3).

$$\begin{vmatrix} 1 & -1 & 0 \\ 3 & 0 & 2 \\ k & 1 & 1 \end{vmatrix} = 0$$

We find that $k = \frac{1}{2}$.

<u>Example 2</u>.

(a) Find an equation that represents a line through the intersection of lines with equations $x + 3y - 3 = 0$ and $2x - 3y - 6 = 0$.

(b) Find an equation of the member of this family of lines

(1) that has slope equal to $\frac{3}{2}$.

(2) that contains the point $(0,3)$.

Solution.

(a) Using Equation (4) we write $m(x + 3y - 3) + n(2x - 3y - 6) = 0$, or $(m + 2n)x + (3m - 3n)y + (-3m - 6n) = 0$.

(b) (1) From the last equation in (a) we have an expression for the slope, which we set equal to $\frac{3}{2}$ and simplify.

$$-\frac{m + 2n}{3m - 3n} = \frac{3}{2}$$

$$-2m - 4n = 9m - 9n$$

$$-11m + 5n = 0$$

We let $m = 5$, $n = 11$, and substitute these values in the equation in (a).

$$27x - 18y - 81 = 0$$

Or, more simply, $3x - 2y - 9 = 0$.

(2) If the line is to contain the point $(0,3)$, these coordinates must satisfy the first equation in (a), therefore

$$m(0 + 9 - 3) + n(0 - 9 - 6) = 0 .$$

Simplifying, we have

$$6m - 15n = 0 .$$

We let $m = 5$, $n = 2$, and obtain

$$x + y - 3 = 0$$

as an equation of the desired line.

Exercises D-3

1. Are the lines with the given equations concurrent? If so, what is their common point?

   (a) $2x - 3y + 6 = 0$, $3x + 4y - 12 = 0$, $x - 4 = 0$
   (b) $x + y - 3 = 0$, $3x - y + 1 = 0$, $2x - 1 = 0$
   (c) $x - y = 4$, $y = x + 7$, $3x - 3y + 5 = 0$

2. For each of the following, determine a real number k such that the equations represent concurrent lines.

   (a) $x - 3y -$ [illegible] $= 0$, $3x + y + 5 = 0$, $kx - 3y - 2 = 0$
   (b) $x + ky - 3 = 0$, $kx - 7y - 6 = 0$, $2x - y - 3k = 0$

3. Given lines L_1, L_2 with equations $3x - 2y + 5 = 0$ and $x + 4y - 1 = 0$; write an equation that represents any line through the point of intersection of L_1 and L_2. Then find the member of this family of lines that

(a) has the slope $\frac{3}{4}$.

(b) is perpendicular to L_1.

(c) contains the origin.

(d) contains the point $(5,2)$.

(e) has a y-intercept of 1.

4. Find an equation of the line parallel to the line whose equation is $3x - y + 7 = 0$, and containing the point of intersection of the lines whose equations are $5x - y + 3 = 0$ and $x + y - 2 = 0$.

5. Given the triangle determined by points $A = (a,0)$, $B = (0,b)$, $C = (c,0)$.

(a) Show that the medians are concurrent, and find their point of intersection. (This point is called the centroid. It was discussed and a vector proof of concurrency given in Example 2, Section 3-8.)

(b) Show that the altitudes are concurrent, and find their point of intersection. (This point is called the orthocenter.)

(c) Show that the perpendicular bisectors of the sides are concurrent, and find their point of intersection. (This point is called the circumcenter; it is the center of the circumscribed circle of the triangle.)

(d) Show that the centroid, the orthocenter, and the circumcenter of this triangle are collinear.

(e) Do you think that what you have proved for triangle ABC is true for any triangle? Give reasons for your answer.

6. Prove that, in a trapezoid, the diagonals and the line drawn through the midpoints of the parallel sides meet in a point.

D-4. Intersections and Parallelism

If two sets have at least one member in common they are said to intersect. We consider in this section, points, lines and planes, and their possible intersections. If set S is a subset of set T, then their intersection is all of S, and we sometimes say that S lies on, or in, T, or S is em-

bedded in T. Thus a point may lie on a line, or a line may be embedded in a plane. Our analytic representations of these sets makes it possible to develop simple criteria for many of these relationships.

Point and Point: P_1 , P_2 : This case is easy to analyze but a good place to start. Two points intersect if and only if they coincide. Their analytic representations are simply their coordinates, which must be identical or equivalent in accordance with the definition of equivalence given when the coordinate systems were introduced.

In rectangular coordinates $P = (3,5)$ differs from $P = (5,3)$. In polar coordinates $P = (6,\pi)$ is the same as $P = (-6,0)$ and $P = (6,3\pi)$.

Point and Line: P_1' , L . A point is on a line if and only if a set of coordinates of the point satisfies an equation of the line. The point $P_1 = (x_1,y_1)$ lies on the line $L : ax + by + c = f(x,y) = 0$, if and only if $f(x_1,y_1) = 0$. The point $P_1 = (x_1,y_1)$ lies on $L : x = a + \ell t$, $y = b + mt$, if and only if there is some value of t , say t_1 , such that $x_1 = a + \ell t_1$, and $y_1 = b + mt_1$. If P_1 and L had been given relative to a polar coordinate system, the discussion would require simple modifications, which are left to the student. The extension of the discussion to 3-space can also be made, with minor revisions which are also left to the student.

Examples.

(a) $P = (1,3)$ is on $L : 3x - 2y + 3 = 0$, because $3(1) - 2(3) + 3 = 0$.
(b) $P = (1,4)$ is not on $L : x = 3 + t$, $y = 2 - 3t$, because the equations $1 = 3 + t$, $4 = 2 - 3t$ impose contradictory conditions on t .
(c) $P = (12,60^\circ)$ is on $L : r = \frac{6}{\cos\theta}$, because $12 = \frac{6}{\cos 60^\circ}$.

Similarly, $Q = (6\sqrt{2}, \frac{\pi}{4})$ and $R = (12,-60^\circ)$ are also on L .
(d) $P = (2,5,-1)$ is on L : $x = 3 + t$, $y = 2 - 3t$, $z = 1 + 2t$, since the equation $2 = 3 + t$ gives a value for t , namely $t = -1$, which is consistent with the equations: $5 = 2 - 3t$ and $-1 = 1 + 2t$.

Point and Plane: P_1 , M . The discussion is left to the student, who is referred to the paragraph above.

Line and Line: L_1 , L_2 . 2-space. Two lines in the same plane may have (1) just one, or (2) all, or (3) no points in common. If the lines are $L_1 : a_1x + b_1y + c_1 = f_1(x,y) = 0$, and $L_2 : a_2x + b_2y + c_2 = f_2(x,y) = 0$, the analytic counterparts of these 3 cases are presented below. Proofs, which are not difficult, are left to the student.

(1) L_1 , L_2 intersect in just one point if and only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

(2) L_1 , L_2 coincide if and only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 0$$

Note that if any two of these determinants are equal to zero, so is the third. Note also, that if this condition is satisfied, there is a non-zero number, k , such that $f_1(x,y) = kf_2(x,y)$.

(3) L_1 , L_2 are parallel if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$,

and either $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ or $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \neq 0$.

(a) Note that, if either of these is different from zero, so is the other.

(b) Note that, for any numbers p and q , the equation $pf_1(x,y) + qf_2(x,y) = 0$ is, in general, an equation of a line, L_3 . If L_1 , L_2 intersect, then L_3 will go through that point of intersection; if L_1 , L_2 coincide, then L_3 will coincide with them; and if L_1 and L_2 are parallel, then L_3 will be parallel to both of them.

If equations for L_1 and L_2 had been presented in parametric or vector form, then the analytic representations of the three cases above would have a somewhat different appearance. The development of these representations is called for in one of the exercises at the end of this section.

3-space. Two lines in 3-space may have (1) just one point in common, (2) all points in common, or no points in common. In 2-space, this last condition requires that the lines be parallel, but in 3-space, lines that have no point in common may be (3) parallel, if they lie in one plane, or (4) skew, if they do not.

The discussion of the first three cases is analogous to the corresponding discussion of the lines in 2-space, but the equations are more complicated.

Suppose L_1 goes through $P_1 = (a_1,b_1,c_1)$ with direction numbers (ℓ_1,m_1,n_1), and L_2 through $P_2 = (a_2,b_2,c_2)$ with direction numbers (ℓ_2,m_2,n_2). Therefore we have equations $L_1 : x = a_1 + \ell_1 s$, $y = b_1 + m_1 s$, $z = c_1 + n_1 s$; and $L_2 : x = a_2 + \ell_2 t$, $y = b_2 + m_2 t$, $z = c_2 + n_2 t$.

(1) If L_1, L_2 intersect at a unique point $P' = (x',y',z')$, there must be values of the parameters, say s' and t', such that

$$x' = a_1 + \ell_1 s' = a_2 + \ell_2 t'$$
$$y' = b_1 + m_1 s' = b_2 + m_2 t'$$
$$z' = c_1 + n_1 s' = c_2 + n_2 t' .$$

These are three linear equations in s' and t', which we may write:

$$\ell_1 s' - \ell_2 t' = a_2 - a_1$$
$$m_1 s' - m_2 t' = b_2 - b_1$$
$$n_1 s' - n_2 t' = c_2 - c_1 .$$

There is a unique common solution if and only if there is a unique solution to any two of these equations which will also satisfy the third. The solution, if any, for the first two equations, say, is:

$$s' = \frac{\begin{vmatrix} a_2 - a_1 & -\ell_2 \\ b_2 - b_1 & -m_2 \end{vmatrix}}{\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix}} , \quad t' = \frac{\begin{vmatrix} \ell_1 & a_2 - a_1 \\ m_1 & b_2 - b_1 \end{vmatrix}}{\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix}} .$$

(Note that these solutions require $\begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} \neq 0$.) The corresponding requirements that there be unique solutions for any two of the above three equations are

$$\begin{vmatrix} \ell_1 & \ell_2 \\ n_1 & n_2 \end{vmatrix} \neq 0 \text{, and } \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \neq 0 .$$

If the s', t' values found above are substituted in the third equation we have:

$$\frac{n_1\begin{vmatrix} a_2 - a_1 & -\ell_2 \\ b_2 - b_1 & -m_2 \end{vmatrix}}{\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix}} - \frac{n_2\begin{vmatrix} \ell_1 & a_2 - a_1 \\ m_1 & b_2 - b_1 \end{vmatrix}}{\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix}} = c_2 - c_1 ;$$

therefore,

$$n_1\begin{vmatrix} a_2 - a_1 & -\ell_2 \\ b_2 - b_1 & -m_2 \end{vmatrix} - n_2\begin{vmatrix} \ell_1 & a_2 - a_1 \\ m_1 & b_2 - b_1 \end{vmatrix} - (c_2 - c_1)\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix} = 0 .$$

This may, after some algebraic juggling, be written in the form

$$(a_2 - a_1)(m_1n_2 - m_2n_1) - (b_2 - b_1)(\ell_1n_2 - \ell_2n_1) + (c_2 - c_1)(\ell_1m_2 - \ell_2m_1) = 0 ;$$

and this in turn may be written in determinant form:

$$\Delta = \begin{vmatrix} a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0 .$$

Note that the elements of the rows are direction numbers for $\overleftrightarrow{P_1P_2}$, L_1, L_2.

(2) , (3) If L_1 and L_2 are parallel or coincident, their direction numbers are equivalent, and all the second order minors of the last two rows must equal zero, and therefore Δ must equal zero. If L_1 and L_2 coincide, they coincide also with $\overleftrightarrow{P_1P_2}$, whose direction numbers must be equivalent to those of L_1 and L_2 , and in that case <u>all</u> the second order minors of Δ must equal zero. If L_1 and L_2 are parallel, then they both intersect $\overleftrightarrow{P_1P_2}$ whose direction numbers may <u>not</u> be equivalent to those of L_1 and L_2 , and in that case the second order minors of Δ which include members from the first row may <u>not</u> all equal zero.

(4) Finally, if L_1 and L_2 are skew , $\Delta \neq 0$.

<u>Example</u>. Consider the lines

$$L_1 : x = 2 + 3t \ , \ y = 3 - t \ , \ z = 4 + 5t \ ,$$
$$L_2 : x = 2 + 2t \ , \ y = -1 + t \ , \ z = 0 + 3t \ ,$$
$$L_3 : x = 3 + 6t \ , \ y = 2 - 2t \ , \ z = 1 + 10t \ ,$$
$$L_4 : x = -1 + 9t \ , \ y = 4 - 3t \ , \ z = -1 + 15t \ .$$

(a) For L_1 and L_2 , $\Delta = \begin{vmatrix} 0 & -4 & -4 \\ 3 & -1 & 5 \\ 2 & 1 & 3 \end{vmatrix} = -24 \neq 0$. $\therefore$ L_1 and L_2 are skew.

(b) For L_1 and L_3 , $\Delta = \begin{vmatrix} 1 & -1 & -3 \\ 3 & -1 & 5 \\ 6 & -2 & 10 \end{vmatrix} = 0$. $\therefore$ L_1 and L_3 are not skew but may intersect in just one point or be parallel or coincident. However,

$$\begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix} = \begin{vmatrix} -1 & 5 \\ -2 & 10 \end{vmatrix} = 0 \ , \quad \therefore L_1 \text{ and } L_3 \text{ cannot}$$

intersect in just one point, but must be parallel or coincident. Coincidence requires all second order minors of Δ to equal zero, and, since

$$\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} = 2 \neq 0 \text{ , the lines are not coincident}$$

and must be parallel.

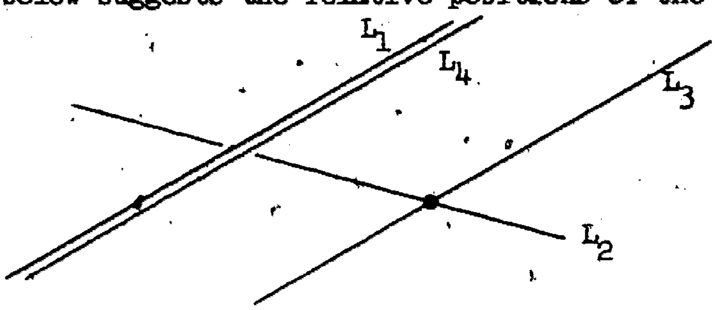
(c) For L_1 and L_4, $\Delta = \begin{vmatrix} -3 & 1 & -5 \\ 3 & -1 & 5 \\ 9 & -3 & 15 \end{vmatrix} = 0$, and also all the second order minors of Δ equal zero. Therefore L_1 and L_4 coincide.

(d) For L_2 and L_3, $\Delta = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 3 \\ 6 & -2 & 10 \end{vmatrix} = 0$, $\therefore L_2$ and L_3 are not skew, but may intersect in just one point, or be parallel or coincident. These last two possibilities are eliminated by the fact that

$$\begin{vmatrix} 2 & 1 \\ 6 & -2 \end{vmatrix} = -10 \neq 0 \text{ , and } \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5 \neq 0 \text{ .}$$

Therefore L_2 and L_3 intersect in just one point, which can be found by the methods in the section above to be $P(6,1,6)$.

The sketch below suggests the relative positions of the four lines.



Exercise. Show analytically that

(a) L_2 and L_4 are skew.

(b) L_3 and L_4 are parallel.

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__Line__ __and__ __Plane__: L , M . A line may (1) be parallel to a plane, (2) be embedded in a plane, or (3) intersect a plane in just one point. In this last case we sometimes say that the line pierces the plane. We develop the analytic counterparts of these three cases.

(1) Suppose L goes through $P_0 = (a_0, b_0, c_0)$ with direction numbers (ℓ_0, m_0, n_0) ; then equations for L are $L : x = a_0 + \ell_0 t$, $y = b_0 + m_0 t$, $z = c_0 + n_0 t$. Suppose we have the plane

$M : px + qy + rz + s = f(x,y,z) = 0$.

Then L will be parallel to M if and only if __no__ point of L lies in M, that is, if there is __no__ value of t such that $p(a_0 + \ell_0 t) + q(b_0 + m_0 t) + r(c_0 + n_0 t) + s = 0$. This is an equation in t , which may be written

$$(pa_0 + qb_0 + rc_0 + s) + (p\ell_0 + qm_0 + rn_0)t = 0 .$$

The coefficient of t resembles the algebraic form of the inner product of two vectors. (See Section 3-5) It is convenient to borrow the algebraic symbolism of vectors and represent this coefficient as the "inner product" of the "vectors" $[p,q,r]$ and $[\ell_0, m_0, n_0]$ With this symbolism, the above equation becomes,

$$f(a_0, b_0, c_0) + [p,q,r] \cdot [\ell_0, m_0, n_0]t = 0 .$$

For this linear equation in t to have no solution, it is necessary and sufficient that __both__: $f(a_0, b_0, c_0) \neq 0$, and $[p,q,r] \cdot [\ell_0, m_0, n_0] = 0$, which are the conditions for L to be parallel to M . These may be recognized as requiring that P_0 , which is a point of L , not lie in M ; and that L be perpendicular to a normal line of M , as established earlier.

__Example__. Show that $L : x = 3 + 2t$, $y = 4 - t$, $z = 1 + 3t$, is parallel to $M : 3x + 3y - z - 5 = f(x,y,z) = 0$.

__Solution__. The criteria developed in the text are satisfied, since:

(1) $f(3,4,1) = 9 + 12 - 1 - 5 = 15 \neq 0$, and

(2) $[2,-1,3] \cdot [3,3,-1] = 2(3) - 1(3) + 3(-1) = 6 - 3 - 3 = 0$.

We might also substitute, in the equation of M , the expressions for x , y , z as functions of t , and get $3(3 + 2t) + 3(4 - t) - (1 + 3t) - 5 = 0$, which leads to the contradiction $15 = 0$. Therefore L doesn't intersect M .

The x-axis, or any line parallel to it, has equations: $x = a_0 + \ell_0 t$, $y = b_0$, $z = c_0$, with direction numbers $(\ell_0, 0, 0)$. If a plane has an equation such as $M : qy + rz + s = 0$, its normal lines have direction numbers $(0,q,r)$. $\therefore$ M is parallel to the x-axis or contains it, since $[\ell_0,0,0] \cdot [0,q,r] = 0$.

In the same way, if a plane has an equation in general form in which the y term is missing, then the plane is parallel to, or contains the y-axis, and so on.

(2) If a line is embedded in a plane, then coordinates of <u>every</u> point of the line must satisfy an equation of the plane. If L and M are given as before: $L : x = a_0 + \ell_0 t$, $y = b_0 + m_0 t$, $z = c_0 + n_0 t$, and $M : px + qy + rz + s = f(x,y,z) = 0$, then this requirement is met if, for all t, $p(a_0 + \ell_0 t) + q(b_0 + m_0 t) + r(c_0 + n_0 t) + s = 0$. This may be written as: $(pa_0 + qb_0 + rc_0 + s) + (p\ell_0 + qm_0 + rn_0)t = 0$, or as: $f(a_0,b_0,c_0) + [p,q,r] \cdot [\ell_0,m_0,n_0]t = 0$.
If this expression is to equal zero for all values of t then we must have: $f(a_0,b_0,c_0) = 0$ and $[p,q,r] \cdot [\ell_0,m_0,n_0] = 0$.

These conditions for embedding may be recognized as requiring that $P_0 = (a_0,b_0,c_0)$, which is a point of L, also be a point of M; and also that L, with direction number (ℓ_0,m_0,n_0) be perpendicular to a normal to M. We have previously used the fact that such a normal has direction numbers (p,q,r).

<u>Example</u>. Show that $L : x = 3 + 2t$, $y = 1 + t$, $z = 3 - t$, lies wholly in $M : 2x - 3y + z - 6 = f(x,y,z) = 0$.

<u>Solution</u>. Both conditions in the section above are satisfied, since

(a) the point $(3,1,3)$ is on M, since $f(3,1,3) = 0$, and

(b) a normal to M has direction numbers $(2,-3,1)$; and L is perpendicular to such a normal, since $[2,-3,1] \cdot [2,1,-1] = 4 - 3 - 1 = 0$.

(3) If we suppose L and M given as in the two cases above, then, if L intersects M in just one point, there must be a unique value of t, say t', such that $P' = (x',y',z')$ on L is also on M. That is, if $x' = a_0 + \ell_0 t'$, $y' = b_0 + m_0 t'$, $z' = c_0 + n_0 t'$, then $p(a_0 + \ell_0 t') + q(b_0 + m_0 t') + r(c_0 + n_0 t') + s = 0$. This is a linear equation in t' which may be written:

$$(pa_0 + qb_0 + rc_0 + s) + (p\ell_0 + qm_0 + rn_0)t' = 0, \text{ or}$$

$$f(a_0,b_0,c_0) + [p,q,r] \cdot [\ell_0,m_0,n_0]t' = 0 .$$

A unique solution will exist if and only if the coefficient of t' is different from zero, that is, $[p,q,r] \cdot [\ell_0,m_0,n_0] \neq 0$. If this condition is satisfied, we may find the unique value of t':

$$t' = -\frac{f(a_0,b_0,c_0)}{[p,q,r] \cdot [\ell_0,m_0,n_0]} .$$

With this value of t' we find the coordinates of P', the unique point of intersection of L and M.

Example. Find the point in which L : $x = 3 + 2t$, $y = 2 - 3t$, $z = 1 + t$, intersects M : $2x - 3y + 4z - 5 = f(x,y,z) = 0$.

Solution. Either by direct substitution of expressions for x, y, z, in equations of L into the equation of M, or by application of the formula above, we obtain:

$$t' = -\frac{f(3,2,1)}{[2,-3,4] \cdot [2,-3,1]} = -\frac{2(3) - 3(2) + 4(1) - 5}{2(2) - 3(-3) + 4(1)}$$

$$t' = -\left(\frac{-1}{17}\right) = \frac{1}{17} . \therefore P' = \left(\frac{53}{17}, \frac{31}{17}, \frac{18}{17}\right) .$$

We may summarize the development in this section so far by observing that much of the analysis depends on the possiblity and nature of the solution of $f(a_0,b_0,c_0) + [p,q,r] \cdot [\ell_0,m_0,n_0]t = 0$. We exhibit the results of our

analysis in the table below.

Case	$f(a_0,b_0,c_0)$	$[p,q,r]\cdot[\ell_0,m_0,n_0]$	numbers of solutions for t
(1) L_1 is parallel to M	$\neq 0$	$= 0$	none
(2) L_1 is embedded in M	$= 0$	$= 0$	infinitely many
(3) L_1 pierces M	any value	$\neq 0$	one

A significant problem, related to the problem of finding the distance between two skew lines, is to find parallel planes which contain two skew lines. Suppose the lines are $L_1 : x = a_1 + \ell_1 t_1$, $y = b_1 + m_1 t_1$, $z = c_1 + n_1 t_1$; and $L_2 : x = a_2 + \ell_2 t_2$, $y = b_2 + m_2 t_2$, $z = c_2 + n_2 t_2$. If the planes M_1 and M_2 are to be parallel, their normals must have equivalent direction numbers, and we may write their equations,

$M_1 : px + qy + rz + s_1 = f_1(x,y,z) = 0$; and

$M_2 : px + qy + rz + s_2 = f_2(x,y,z) = 0$. The problem is to find p , q , r , s_1 , and s_2 in terms of the constants which give us L_1 and L_2 , under the conditions imposed by the problem. Since L_1 and L_2 are embedded respectively in M_1 and M_2 , we have from the previous section ,

$f_1(a_1,b_1,c_1) = f_2(a_2,b_2,c_2) = 0$, and also

$[p,q,r]\cdot[\ell_1,m_1,n_1] = [p,q,r]\cdot[\ell_2,m_2,n_2] = 0$. These four equations are not sufficient to find the five values p , q , r , s_1 and s_2, but we recognize that direction numbers need not be found uniquely; any equivalent set will do as well, to write equations for M_1 and M_2 . We assume that not all of (p,q,r) equal zero, and, in particular that, say, $r \neq 0$, in which case we have an equivalent set $(\frac{p}{r},\frac{q}{r},1)$; and the algebraic problem of solving four equations in four variables.

The algebraic conditions for solvability have their geometric counterparts, corresponding to the relative positions of L_1 and L_2 . We consider here only the situation in which L_1 and L_2 are skew. The general algebraic treatment of this case involves extensive algebraic manipulation, which we shall not go through. We will carry through the details in an example.

Example. Find parallel planes M_1 and M_2 which contain the lines $L_1 : x = 3 - t_1$, $y = 2 + 3t_1$, $z = 1 + 2t_1$; and $L_2 : x = -2 + 3t_2$, $y = 3 + 2t_2$, $z = 1 - 2t_2$.

Solution.

(1) L_1 is not parallel to L_2 , because their direction numbers are not equivalent.

(2) L_1 and L_2 do not meet, because the assumption of a common point imposes contradictory conditions on t_1 and t_2 . If we try to solve the system

$$\begin{cases} 3 - t_1 = -2 + 3t_2 \\ 2 + 3t_1 = 3 + 2t_2 \\ 1 + 2t_1 = 1 - 2t_2 , \end{cases}$$

the last two equations require $t_1 = \frac{1}{5}$, $t_2 = -\frac{1}{5}$, and these do not satisfy the first equation.

(3) Therefore, L_1 and L_2 are skew. Then, as in the section above, we consider planes $M_1 : px + qy + rz + s_1 = f_1(x,y,z) = 0$, and $M_2 : px + qy + rz + s_2 = f_2(x,y,z) = 0$. The conditions that L_1 and L_2 be perpendicular to a common normal N to planes M_1 and M_2 , become:

$$\begin{cases} (-1)p + 3(q) + 2(r) = 0 \\ (3)p + 2(q) - 2(r) = 0 . \end{cases}$$

We may rewrite these as

$$\begin{cases} -1(\frac{p}{r}) + 3(\frac{q}{r}) + 2 = 0 , \\ 3(\frac{p}{r}) + 2(\frac{q}{r}) - 2 = 0 , \end{cases}$$

and these yield, by elementary methods, the solutions, $\frac{p}{r} = \frac{10}{11}$, $\frac{q}{r} = \frac{-4}{11}$. We may therefore use either the direction numbers

$(\frac{10}{11}, \frac{-4}{11}, 1)$ or the equivalent $(10,-4,11)$. With these values of

p, q, r, we find s_1 and s_2 easily from the conditions that M_1 and M_2 contain points $P_1 = (3,2,1)$ and $P_2 = (-2,3,1)$ of L_1 and L_2 respectively, i.e.

$$p(3) + q(2) + r(1) + s_1 = 0 , \quad \therefore s_1 = -33 ,$$

$$p(-2) + q(3) + r(1) + s_2 = 0 , \quad \therefore s_2 = 21 .$$

Finally we have the equations of the planes

$M_1 : 10x - 4y + 11z - 33 = 0$; $M_2 : 10x - 4y + 11z + 21 = 0$.

Two Planes: M_1, M_2. Suppose these planes have respective equations:

$$M_1 : p_1x + q_1y + r_1z + s_1 = f_1(x,y,z) = 0 ,$$

$$M_2 : p_2x + q_2y + r_2z + s_2 = f_2(x,y,z) = 0 .$$

The planes may (1) coincide, (2) be parallel, or (3) intersect.

(1) The planes coincide if and only if every point of one of them is a point of the other, and this will be the case if and only if there is some non-zero number k such that $f_1(x,y,z) = kf_2(x,y,z)$, as may be easily seen.

(2) The planes will be parallel if and only if they have a common normal, but no common point. These conditions will both be met if there is a number $k \neq 0$, such that $p_1 = kp_2$, $q_1 = kq_2$, $r_1 = kr_2$ but $s_1 \neq ks_2$. The proof that this is so is left to the student.

(3) If two distinct planes intersect in a point $P_0 = (x_0,y_0,z_0)$, one of the earlier postulates of geometry requires that they intersect in a line containing P_0. We show, from the analytic representation and condition that this is so, and find the line, given the planes.

The general treatment would involve tedious computation, and would probably not be as enlightening as a specific example.

Example. Suppose two planes, $M_1 : 2x - 3y + z - 4 = f_1(x,y,z) = 0$, and $M_2 : x + 2y - 4z - 1 = f_2(x,y,z) = 0$, have the point $P_0 = (3,1,1)$ in common. Show that they have in common a line containing P_0.

Solution. If p and q are numbers not both zero, the equation $pf_1(x,y,z) + qf_2(x,y,z) = 0$ is, in general, an equation of a plane containing P_0. This equation may be written as:

$$(2p + q)x + (-3p + 2q)y + (p - 4q)z + (-4p - q) = 0 .$$

If, in particular, $p = 1$, $q = -2$, this equation becomes $-7y + 9z - 2 = 0$ or $7y - 9z + 2 = 0$. The locus, in 3-space of this equation is, as shown in the previous section, a plane, parallel to the x-axis. Note that this plane contains $P_0 = (3,1,1)$, since $7(1) - 9(1) + 2 = 0$. If we subtract corresponding members of these two equations we get, as another equation of this plane, $7(y - 1) - 9(z - 1) = 0$.

In the same way, by taking $p = 2$, $q = 3$, we get the equation $7x - 10z - 11 = 0$, which represents a plane parallel to the y-axis, and also containing $P_0 = (3,1,1)$, since $7(3) - 10(1) - 11 = 0$. If we subtract corresponding members of these two equations we get $7(x - 3) - 10(z - 1) = 0$. These equations of the two planes parallel to the x- and y-axes, respectively, may be written:

$$\frac{y - 1}{9} = \frac{z - 1}{7},$$

$$\frac{x - 3}{10} = \frac{z - 1}{7}.$$

Note that these three fractional expressions are all equal and can be set equal to some common value t, from which we get $x = 3 + 10t$, $y = 1 + 9t$, and $z = 1 + 7t$.

These are clearly a set of parametric equations for a line L containing the point $(3,1,1)$. To show that L lies wholly in M_1 we must show, that for all values of t,

$$2(3 + 10t) - 3(1 + 9t) + 1(1 + 7t) - 4 = 0 ,$$

that is,

$$6 + 20t - 3 - 27t + 1 + 7t - 4 = 0 ,$$

which becomes, for all t, $0 = 0$.

In the same way, to show that L lies wholly in M_2, we must show, that for all values of t,

$$1(3 + 10t) + 2(1 + 9t) - 4(1 + 7t) - 1 = 0 ,$$

that is,

$$3 + 10t + 2 + 18t - 4 - 28t - 1 = 0 ,$$

and, for all t, this becomes, $0 = 0$.

Exercises D-4

Consider the four lines given by the equations below for Exercises 1 to 6.

$$L_1 : \begin{cases} x = -2 + 3t_1 \\ y = 3 - t_1 \\ z = 4 - 2t_1 \end{cases} \quad L_2 : \begin{cases} x = 3 - 6t_2 \\ y = -5 + 2t_2 \\ z = 1 + 4t_2 \end{cases} \quad L_3 : \begin{cases} x = -5 + 3t_3 \\ y = 6 - 2t_3 \\ z = 13 - 8t_3 \end{cases}$$

$$L_4 : \begin{cases} x = 7 + 3t_4 \\ y = -6 + 4t_4 \\ z = 9 - 6t_4 \end{cases}$$

1. Determine for each pair below if the lines (a) intersect in just one point, or (b) are parallel, or (c) are coincident, or (d) are skew. If a pair intersect in just one point, find that point.

   (a) L_1, L_2 (d) L_2, L_3

   (b) L_1, L_3 (e) L_2, L_4

   (c) L_1, L_4 (f) L_3, L_4

2. Write an equation for the line which contains $P = (1,2,3)$ and is parallel to

   (a) L_1. (c) L_3.

   (b) L_2. (d) L_4.

3. Write equations of parallel planes M_1 and M_2 which contain respectively

   (a) L_1 and L_3. (b) L_2 and L_4.

4. Write an equation of a plane which

   (a) contains L_1 and is parallel to L_3.

   (b) contains L_4 and is parallel to L_1.

5. Write an equation for the plane which contains the origin and

   (a) L_1. (c) L_3.

   (b) L_2. (d) L_4.

L_A is said to go over L_B if L_A and L_B are disjoint (have no point in common), and there is a point P_A on L_A which is above a point P_B on L_B; that is, such that $x_A = x_B$, $y_A = y_B$ and $z_A > z_B$. There is a

corresponding definition for a line to go under another line. We show that L_1 goes over L_3 because if $x_1 = x_3$ and $y_1 = y_3$ we have $-2 + 3t_1 = -5 + 3t_3$, and $3 - t_1 = 6 - 2t_3$, therefore $t_1 = 1$ and $t_3 = 2$. For these values of t_1 and t_3 we have $z_1 = 2$ and $z_3 = -3$, $z_1 > z_3$, and therefore L_1 goes over L_3.

6. Determine the over or under relationship for these pairs of lines:

   (a) L_1 and L_4 (c) L_2 and L_4

   (b) L_2 and L_3 (d) L_3 and L_4

7. If L_A goes over L_B, and L_B goes over L_C, is it always, sometimes, or never true that L_A goes over L_C?

8. True or false? One of two disjoint lines is over the other. Explain.

Consider the four planes $M_1 : 3x - 2y + z - 5 = 0$, for Exercises 9 to 12. $M_2 : 2x + y - 3z + 4 = 0$; $M_3 : x + 3y - 2z - 1 = 0$, and $M_4 : -2x + y + 2z + 3 = 0$.

9. Find in parametric form, equations of the line of intersection of

   (a) M_1, M_2. (d) M_2, M_3.

   (b) M_1, M_3. (e) M_2, M_4.

   (c) M_1, M_4. (f) M_3, M_4.

10. Find the common intersection point, if any, of

    (a) M_1, M_2, M_3. (c) M_1, M_3, M_4.

    (b) M_1, M_2, M_4. (d) M_2, M_3, M_4.

    Note that we may use the results of Exercise 9 to facilitate the computation in Exercise 10.

11. Write an equation of the plane which contains the origin, and is parallel to:

    (a) M_1 (c) M_3

    (b) M_2 (d) M_4

12. (Refer to the lines at the top of this group of exercises.) Find the point, if any, in which

(a) L_1 meets M_1.   (c) L_3 meets M_3.

(b) L_2 meets M_2.   (d) L_4 meets M_4.

13. Suppose equations of two lines in 2-space are given in parametric form. Develop criteria, in terms of the constants in these equations, for the various geometric relationships that may exist between the lines, as in Section 4-6D, where the equations were given in general form.

D-5. Perpendicularity and Angles between Lines and Planes

We have used quite freely in this chapter the definitions and tests for perpendicularity that had been developed in Chapter 2. For the purposes of this chapter we consider angles between lines and planes in general, and perpendicularity as the special relationship that exists when these angles are right angles. We recall that an angle has been defined as the union of two non-collinear rays with a common end-point.

Two lines: L_1, L_2. We do not define angles between parallel or coincident lines. Although there may be some value in the consideration of "straight angles", or "zero angles", we feel that there is not sufficient application of these concepts in this text to warrant the time and effort that their treatment would entail. We have already developed in earlier sections analytic criteria to distinguish cases of parallelism or coincidence.

If L_1 and L_2 are neither parallel nor coincident we define the angles between them to be the angles formed by lines L'_1 and L'_2 which contain some common point, say, the origin, and are respectively coincident with or parallel to L_1 and L_2. Note that this definition covers any intersecting or skew lines. Such lines determine four angles, which can be analytically distinguished only if there is some way of establishing, implicitly or explicitly, a sense on L_1 and L_2.

2-space: Consider the intersecting lines $L_1 : x = a_1 + \lambda_1 t$, $y = b_1 + \mu_1 t$, and $L_2 : x = a_2 + \lambda_2 t$, $y = b_2 + \mu_2 t$, where λ_1, μ_1, λ_2, μ_2, are direction cosines. Then the lines L'_1 and L'_2 which go through

the origin and are respectively parallel to or coincident with L_1 and L_2 have the equations:

$$L'_1 : x = \lambda_1 t \ , \ y = \mu_1 t ; \quad L'_2 : x = \lambda_2 t \ , \ y = \mu_2 t \ .$$

Note that λ_1, μ_1 establish a sense along L_1 and L'_1; the "positive" part containing points for which $t > 0$; and so on. If, on L'_1 and L'_2, we take $t = 1$, we get the points $P_1 = (\lambda_1, \mu_1)$ and $P_2 = (\lambda_2, \mu_2)$ on the positive rays $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$. We define the angle between L_1 and L_2 as given above, to be the angle formed by $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$, which we designate as θ. Note that if we had taken for L_1 the equivalent direction cosines $-\lambda_1$, $-\mu_1$, these would have been established on L_1 a sense opposite to the original, and in that case the angle between L_1 and L_2 would have been the supplement of θ. It is not difficult to see that, for any choices of equivalent direction cosines for L_1 and L_2 the angle between L_1 and L_2 would be congruent either to θ or its supplement. These are the angles we mean when we speak of the angles formed by two lines.

From ΔOP_1P_2 and the law of cosines we get

$d^2(P_1,P_2) = d^2(O,P_1) + d^2(O,P_2) - 2d(O,P_1)d(O,P_2) \cos \theta$. Note that

$$d(O,P_1) = d(O,P_2) = 1 \text{ , and } d^2(P_1,P_2) = (\lambda_1 - \lambda_2)^2 + (\mu_1 - \mu_2)^2$$

$$= \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 + \mu_1^2 - 2\mu_1\mu_2 + \mu_2^2$$

$$= 2 - 2\lambda_1\lambda_2 - 2\mu_1\mu_2 \ .$$

Therefore $\quad 2 - 2\lambda_1\lambda_2 - 2\mu_1\mu_2 = 2 - 2 \cos \theta$

(1) and $\quad \cos \theta = \lambda_1\lambda_2 + \mu_1\mu_2 \ .$

This is an unambiguous determination for <u>one</u> of the angles between L_1 and L_2, namely that between the positive rays on L_1 and L_2 determined by the given direction cosines and $t > 0$. Another of the angles between L_1 and L_2 is clearly the supplement of θ.

Note that $L_1 \perp L_2$ if and only if the angles between them are right angles, that is, if and only if $\lambda_1\lambda_2 + \mu_1\mu_2 = 0$. This is a familiar criterion for perpendicularity.

We may indicate the corresponding results using direction numbers, rather than direction cosines. Note that when we set

$$\lambda = \frac{\ell}{\pm\sqrt{\ell^2 + m^2}}, \quad \mu = \frac{m}{\pm\sqrt{\ell^2 + m^2}}$$

there is an ambiguity introduced with the choice of sign for the radical. A particular pair of direction numbers entails an implicit sensing of the line, as with the case of direction cosines; the positive sign for both radicals preserves the original sensing. In terms of direction numbers, Equation (1) becomes

$$(2) \qquad \cos\theta = \frac{\ell_1\ell_2 + m_1m_2}{\sqrt{\ell_1^2 + m_1^2}\,\sqrt{\ell_2^2 + m_2^2}},$$

and the corresponding condition for perpendicularity becomes

$$\ell_1\ell_2 + m_1m_2 = 0 .$$

The development here resembles, as it should, the corresponding development with vectors, given in Section 3-7. We may, in these formulas, use the symbolism of vectors, to simplify their representations. We recognize that the vector $\overrightarrow{OP}_1 = [\lambda_1,\mu_1]$ and $\overrightarrow{OP}_2 = [\lambda_2,\mu_2]$. Therefore we may write Equation (1) in vector form:

$$\cos\theta = [\lambda_1,\mu_1]\cdot[\lambda_2,\mu_2] = \overrightarrow{OP}_1 \cdot \overrightarrow{OP}_2 .$$

In the same way, although we have not used vectors whose components are direction numbers, we may extend our symbolism and treat the expression $[\ell,m]$ algebraically as if it were a vector, in which case we may write Equation (2) in "vector" form:

$$\cos\theta = \frac{[\ell_1,m_1]\cdot[\ell_2,m_2]}{\|[\ell_1,m_1]\|\,\|[\ell_2,m_2]\|},$$

and the corresponding condition for perpendicularity as

$$[\ell_1,m_1]\cdot[\ell_2,m_2] = 0 .$$

Example 1. Find the angle between $L_1 : x = 2 + 3t$, $y = 4 - t$, and $L_2 : x = 3 + t$, $y = 2 + 2t$.

Solution.
$$\cos \theta = \frac{(3)(1) + (-1)(2)}{\sqrt{3^2 + (-1)^2}\ \sqrt{1^2 + 2^2}}$$

$$\cos \theta = \frac{3 - 2}{\sqrt{10}\ \sqrt{5}} = \frac{1}{\sqrt{50}} \approx .14$$

$$\therefore \quad \theta \approx 82^\circ$$

Example 2. Show that $L_3 : x = 3 - 5t$, $y = 2 + 3t$ is perpendicular to $L_4 : x = 1 + 3t$, $y = 4 + 5t$.

Solution. $(-5)(3) + (3)(5) = 0$, $\therefore L_3 \perp L_4$.

Example 3. Find the angles between L_1 and L_2 , where L_1 contains the points $(3,4)$, $(-1,-1)$: and L_2 contains the points $(-4,6)$, $(3,0)$.

Solution. Since no sense is imposed on L_1 and L_2 we will find their angles of intersection.

We may take as direction numbers for L_1 , $(4,5)$ and for L_2 , $(-7,6)$. (Why?) Therefore:

$$\cos \theta = \frac{(4)(-7) + (5)(6)}{\sqrt{4^2 + 5^2}\ \sqrt{(-7)^2 + 6^2}} \approx .034$$

$$\therefore \quad \theta \approx 88^\circ .$$

We may, most simply, find the other angle of intersection as the supplement of θ , but it is instructive to use equivalent direction numbers for L_1 which have the effect of reversing the sense induced by the first choice. We use now $(-4,-5)$, and $(-7,6)$ as pairs of direction numbers and get

$$\cos \theta' = \frac{(-4)(-7) + (-5)(6)}{\sqrt{(-4)^2 + (-5)^2}\ \sqrt{(-7)^2 + 6^2}} \approx -.034$$

$$\therefore \quad \theta' \approx 92^\circ$$

which is, as we expected, supplementary to θ .

Example 4. Find the line L_5, to contain the point $(1,2)$ and be perpendicular to $L_1 : x = 2 + 3t$, $y = 4 - t$.

Solution. Suppose L_5 meets L_1 at $P = (a,b)$. Then we take direction numbers for L_5 as $(a - 1, b - 2)$. From the perpendicularity relationship we have $3(a - 1) - 1(b - 2) = 0$. From the fact that $P = (a,b)$ is on L_1, we have $a = 2 + 3t$, $b = 4 - t$. Substituting these expressions for a and b into the first of these three equations yields $3(1 + 3t) - 1(2 - t) = 0$, from which $t = -.1$. Therefore $P = (1.7, 4.1)$ and L has the equations: $x = 1.7 + .7s$, $y = 4.1 + 2.1s$.

Two lines: 3-space. The development here is a straightforward generalization from that given for 2-space. As before, the significant formula comes from the consideration of ΔOP_1P_2, where L_1 and L_2 either contain or are parallel to $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$. The results are indicated below, but the proofs, which are not at all difficult, are left to the student.

$$\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2$$

(3) or
$$\cos\theta = \frac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{\sqrt{\ell_1^2 + m_1^2 + n_1^2}\ \sqrt{\ell_2^2 + m_2^2 + n_2^2}}.$$

As before, the test for perpendicularity becomes

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0, \text{ or } \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0.$$

These may be represented simply, in vector form, as

$$[\lambda_1,\mu_1,\nu_1]\cdot[\lambda_2,\mu_2,\nu_2] = 0, \text{ or } [\ell_1,m_1,n_1]\cdot[\ell_2,m_2,n_2] = 0.$$

Example 1. Find the angle between two lines having direction cosines as follows:

$$\lambda_1 = \frac{-2}{\sqrt{5}},\ \mu_1 = 0,\ \nu_1 = \frac{1}{\sqrt{5}}, \text{ and } \lambda_2 = \frac{1}{\sqrt{3}},\ \mu_2 = \frac{1}{\sqrt{3}},\ \nu_2 = \frac{1}{\sqrt{3}}.$$

Solution.

$$\cos\theta = [-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}]\ [\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$$

$$= -\frac{1}{\sqrt{15}} \approx -.258$$

$$\therefore\ \theta \approx 105^\circ$$

Example 2. Show that the lines $L_1 : x = 2 + 3t$, $y = 3 - t$, $z = 2 + 4t$, $L_2 : x = 5 + t$, $y = 6 + 7t$, $z = 7 + t$, are perpendicular to each other.

Solution. $[3,-1,4] \cdot [1,7,1] = (3)(1) + (-1)(7) + (4)(1) = 0$.

Example 3. Find the line L_3 which contains $P = (7,4,5)$ and is perpendicular to L_1 of the previous exercise.

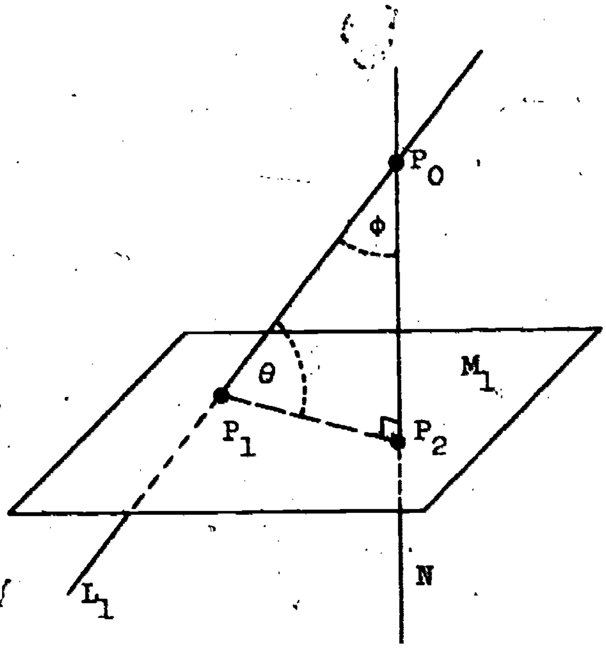
Solution. If L_3 meets L_1 at $P = (a,b,c)$ then we may take, as direction numbers for L_3 , $(a - 7 , b - 4 , c - 5)$. The condition for perpendicularity requires $3(a - 7) - 1(b - 4) + 4(c - 5) = 0$. Since $P = (a,b,c)$ is on L_1 , we have $a = 2 + 3t$, $b = 3 - t$, and $c = 2 + 4t$. If we substitute these expressions for the coordinates into the previous equation we get:

$$3(-5 + 3t) - 1(-1 - t) + 4(-3 + 4t) = 0 \text{, from which } t = 1 .$$

Therefore $P = (5,2,6)$ and L_3 has the equations: $x = 7 + 2t$, $y = 4 + 2t$, $z = 5 - t$.

Line and Plane: L_1 , M_1 . It is convenient to consider the line $L_1 : x = a_1 + \ell_1 t$, $y = b_1 + m_1 t$, $z = c_1 + n_1 t$; and the plane $M_1 : px + qy + rz + s = 0$. We have already developed criteria for L_1 to be parallel, or perpendicular to M_1 . Suppose it is neither, and intersects M_1 at point P_1 . Then any other point of L_1 , say P_0 determines, with M_1 , a unique line N , perpendicular to M_1 , and meeting it at, say, P_2 . We define the angle between L_1 and M_1 to be the angle $P_0P_1P_2$, designated as θ . Note that this definition requires $0^\circ < \theta < 90^\circ$

Since N has direction numbers (p,q,r) and L_1 has direction numbers (ℓ_1,m_1,n_1), we can find the angles between L_1 and N, from Equation (3) of the previous section. We need the acute angle, designated ϕ, and therefore use the absolute value of the right member as $\cos \phi$. But, from right $\Delta P_0P_1P_2$, since θ and ϕ are complementary, we have $\sin \theta = \cos \phi$, and the equation we want:



(4) $$\sin \theta = \frac{|\ell_1 p + m_1 q + n_1 r|}{\sqrt{\ell_1^2 + m_1^2 + n_1^2}\ \sqrt{p^2 + q^2 + r^2}}$$

<u>Example</u>. Find the angle between $L_1 : x = 2 + t$, $y = 3 - 2t$, $z = 1 + t$; and $M_1 : 3x + 4y - 12z + 5 = 0$.

<u>Solution</u>.

$$\sin \theta = \frac{|1(3) - 2(4) + 1(-12)|}{\sqrt{1^2 + (-2)^2 + 1^2}\ \sqrt{3^2 + 4^2 + (-12)^2}} = \frac{|-17|}{\sqrt{6}\ \sqrt{169}};$$

$$\sin \theta = \frac{17}{13\sqrt{6}} \approx .53 \quad \therefore \quad \theta \approx 32^\circ$$

<u>Two planes</u>: M_1, M_2. Consider the planes, $M_1 : p_1x + q_1y + r_1z + s_1 = 0$, $M_2 : p_2x + q_2y + r_2z + s_2 = 0$, and a point $P_0 = (a_0,b_0,c_0)$ not lying in either plane. P_0 and M_1 determine a unique normal line N_1, and P_0 and M_2 a unique normal line N_2. We define the angles between planes M_1 and M_2 to be the angles between lines N_1 and N_2. If N_1 and N_2 coincide, then the planes are perpendicular to a common line and must be parallel or coincident. The analytic conditions are easy to find. Since N_1 and N_2 contain a common point P_0, and have direction numbers (p_1,q_1,r_1) and (p_2,q_2,r_2) they will coincide if and only if these direction numbers are equivalent, that is, if there is a number $k \neq 0$, such that $p_1 = kp_2$, $q_1 = kq_2$,

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$r_1 = kr_2$; and these are the conditions that M_1 be parallel to or coincident with M_2 , as has been noted earlier. Of course M_1 and M_2 will coincide if and only if, further, $s_1 = ks_2$, otherwise M_1 and M_2 are parallel.

If N_1 and N_2 do not coincide, the angles between them can be found from Equation (3) of the previous section, and these are precisely the angles between M_1 and M_2 :

$$(5) \qquad \cos\theta = \frac{p_1p_2 + q_1q_2 + r_1r_2}{\sqrt{p_1^2 + q_1^2 + r_1^2}\ \sqrt{p_2^2 + q_2^2 + r_2^2}}$$

If one of these angles is designated as θ , another must be the supplement of θ , and the remaining two angles congruent to these. Then the right member of Equation (5) gives the cosine either of θ or of its supplement. We are usually interested in the acute angle, in which case we use the absolute value of the right member of (5) .

Example. Find the angles between the planes $M_1 : x - 2y + z - 4 = 0$, and $M_2 : 2x + 2y - z + 3 = 0$.

Solution.

$$\cos\theta = \frac{1(2) - 2(2) + 1(-1)}{\sqrt{1^2 + (-2)^2 + 1^2}\ \sqrt{2^2 + 2^2 + (-1)^2}}$$

$$= \frac{-3}{\sqrt{6}\ \sqrt{9}} \approx -.41$$

$$\theta \approx 156^\circ$$

$\therefore$ The angles are 156° and 24° .

Example. Find an equation of the plane, perpendicular to line $L : x = 2 + t$, $y = 3 - 2t$, $z = 1 + 3t$, and containing the point $A = (3,1,2)$.

Solution. If $P = (x,y,z)$ is any point of the plane, then direction numbers for $\overleftrightarrow{PA}$ are $(x - 3 , y - 1 , z - 2)$. The condition of perpendicularity requires that

$$1(x - 3) - 2(y - 1) + 3(z - 2) = 0 ,$$

and this is the solution, which may be written more compactly as

$$x - 2y + 3z - 7 = 0 .$$

Exercises D-5

Consider these three lines for Exercises 1 to 4.

$$L_1 : x = 3 - t , y = 2 + 3t$$
$$L_2 : x = 2 + t , y = 1 - 2t$$
$$L_3 : x = 1 + 3t , y = 3 + 2t .$$

1. (a) Find the angle between L_1 and L_2.

   (b) Find the angle between L_1 and L_3.

   (c) Find the angle between L_2 and L_3.

2. Find the line through the point $(3,5)$ and perpendicular to

   (a) L_1 (b) L_2 (c) L_3

3. Find the bisectors of the angles formed by L_1 and L_2, using the locus definition of an angle bisector, (points equidistant from the given lines); then show, by the methods of this section, that the angles have been cut into congruent pairs.

4. If L_1, L_2 meet at P_3; L_2, L_3 meet at P_1; and L_3, L_1 at P_2,

   (a) find the coordinates of P_1, P_2, P_3.

   (b) Use these results to find the lines which contain the three altitudes of $\Delta P_1P_2P_3$.

5. At what angles does the line determined by $(1,3)$, $(4,-2)$, meet the line determined by $(-1,2)$, $(2,-3)$?

Consider these lines for Exercises 6 to 14.

$$L_1 : x = 2 - 3t , y = 3 + t , z = 4 + 2t$$
$$L_2 : x = 3 + t , y = 4 - t , z = 2 + 3t$$
$$L_3 : x = 1 + 2t , y = 2 + t , z = 4 - 3t$$

6. Find the angles

   (a) between L_1 and L_2.

   (b) between L_1 and L_3.

   (c) between L_2 and L_3.

7. Find the equations of a line through $P = (1,2,3)$ and perpendicular to

(a) L_1. (b) L_2. (c) L_3.

8. Find equations of a line

(a) N_1 perpendicular to both L_2 and L_3.

(b) N_2 perpendicular to both L_1 and L_3.

(c) N_3 perpendicular to both L_1 and L_2.

9. Find an equation of a plane which contains the point $P = (3,5,7)$ and is perpendicular to

(a) L_1. (b) L_2. (c) L_3.

10. Find an equation of a plane which

(a) contains L_1 and is parallel to L_2.

(b) contains L_1 and is parallel to L_3.

(c) contains L_2 and is parallel to L_1.

(d) contains L_2 and is parallel to L_3.

(e) contains L_3 and is parallel to L_1.

(f) contains L_3 and is parallel to L_2.

Consider these planes

$$M_1 : 2x + 3y - z + 5 = 0$$

$$M_2 : 3x - y + 2z - 4 = 0$$

$$M_3 : x + 2y + 3z + 7 = 0$$

11. Find the angles between

(a) M_1, M_2 (b) M_1, M_3 (c) M_2, M_3

12. Find the plane which

(a) contains L_1 and is perpendicular to M_1.

(b) contains L_1 and is perpendicular to M_2.

(c) contains L_1 and is perpendicular to M_3.

(d) contains L_2 and is perpendicular to M_1.

(e) contains L_2 and is perpendicular to M_2.

(f) contains L_2 and is perpendicular to M_3 .

(g) contains L_3 and is perpendicular to M_1 .

(h) contains L_3 and is perpendicular to M_2 .

(i) contains L_3 and is perpendicular to M_3 .

13. Find the plane which contains the origin, and is perpendicular to the line determined by

(a) M_1 , M_2 (b) M_1 , M_3 (c) M_2 , M_3

14. Find the angles between each of the lines L_1 , L_2 , L_3 , given above, and each of the planes, M_1 , M_2 , M_3 :

(a) L_1M_1 (d) L_2M_1 (g) L_3M_1

(b) L_1M_2 (e) L_2M_2 (h) L_3M_2

(c) L_1M_3 (f) L_2M_3 (i) L_3M_3

15. Find the angle that each axis makes with each plane.

(a) M_1 (b) M_2 (c) M_3

16. Consider two intersecting lines in 2-space, whose equations are
$L_1 : a_1x + b_1y + c_1 = f_1(x,y) = 0$, and
$L_2 : a_2x + b_2y + c_2 = f_2(x,y) = 0$. Develop a formula for the cosine of one of the angles between them, in terms of a_1 , b_1 , c_1 , a_2 , b_2 , c_2 .

Supplement to Chapter 7

Part 1

CONIC SECTIONS

S7-1. Cones and Sections of Cones

In your study of geometry you learned that a circular cone may be defined as the union of all segments $\overline{VP}$ where P is any point contained in a circular region C and V is any point of space not contained in the plane of C. The resulting geometric configuration is a solid. If O is the center of C and if $\overleftrightarrow{OV}$ is perpendicular to the plane of C, the resulting solid is a right circular cone.

An alternative idea of a cone is as an unbounded surface rather than as a bounded solid.

DEFINITIONS. Let D be a curve contained in a plane E and let V be any point not in E. Then the union of all lines $\overleftrightarrow{VP}$ where P is a point of D, is a cone.

The curve D is a plane curve and the directrix of the cone; the point V is the vertex of the cone; the lines $\overleftrightarrow{VP}$ are the elements of the cone. Note that according to this definition of a cone the surface falls naturally into two parts.

DEFINITION. If V is the vertex of a cone, D is the directrix of the cone, and P is any point of D, then the union of the rays $\overrightarrow{VP}$ is a nappe of the cone; the union of the rays opposite to $\overrightarrow{VP}$, is also a nappe of the cone.

It becomes apparent that while a given cone has a unique vertex, it has infinitely many possible directrices.

Cones may be named after curves which are their directrices. Thus a cone which has a circle as a directrix is called a circular cone. The line containing the vertex of the cone and the center of the circle is called the axis of the cone. If the axis of the cone is perpendicular to the plane of the circle, then the cone is called a right circular cone. The right circular cones are the cones which we shall consider. We state two theorems with the proofs suggested as exercises.

THEOREM S7-1. A circular cone is a right circular cone if and only if the points of a directrix are equidistant from the vertex.

THEOREM S7-2. The points of the axis of a right circular cone are equidistant from the elements of the cone.

The intersection of a surface and a plane is called a section of the surface. If the surface is directed or generated by a plane curve (as are cones, prisms, cylinders, and pyramids), then the sections of the surface formed by planes parallel to the plane of the generating curve are called cross-sections of the surface. If the surface has an axis, then the sections of the surface formed by planes perpendicular to the axis are called right-sections. Since the axis of a right circular cone is perpendicular to the plane of the directrix, the cross-sections and right-sections are identical. The sections of a right circular cone are called conic sections. They may also be obtained from other cones and surfaces. This will be made clear in Chapter 9. However, we shall confine our approach here to sections of right circular cones.

What we plan to do is to use geometric methods to discover certain characteristics of the conic sections. These characteristics enable us to use analytic methods to study the conic sections as curves in the intersecting plane.

Exercises S7-1

1. Prove Theorem S7-1.
2. Prove Theorem S7-2.

S7-2. Tangent Spheres and Cutting Planes

Let us consider the sections of a right circular cone. For the time being we shall not consider those sections which contain the vertex of the cone. Such sections are classified as degenerate conic sections and will be studied separately. Let V be the vertex of the cone, a the axis of the cone, and E the intersecting or cutting plane. There are associated with each section one or more spheres with center on the axis a which are tangent both to E and to all the elements of the cone. It is our first task to prove the existence of such a sphere or spheres.

From the definition of a right circular cone, it follows that any two elements of the cone form congruent acute angles with the axis. We define the measure of these acute angles to be the elemental angle of the cone, which we denote by x.

We recall that the distance from a point to a line is the length of a segment which is perpendicular to the line and of which the end points are the given point and a point in the line. Also, the distance from a point to a plane is the length of a segment which is perpendicular to the plane and of which the end points are the given point and a point in the plane.

The axis of the cone is the set of all points which are equidistant from the elements of the cone. We say therefore that each point of the axis is the same distance from the cone and that this distance is the distance between the point and the cone.

Given any real number except zero, there exist two points on the axis which are this measure of distance from the cone, one on either side of the vertex. For the real number zero there exists only one such point, the vertex of the cone. For each of these points on the axis, the points of the cone at the given distance lie in the same plane and form a circle. Since these are the closest points of the cone, there is a sphere with center at the given point and radius equal to the given distance, which is tangent to each element of the cone. For this reason we say that the sphere is tangent to the cone. The union of the points of tangency is a circle, called the circle of tangency.

We turn our attention to the plane intersecting the cone. This plane may be parallel to the axis of the cone, but in all other cases it intersects the axis, either in the axis itself or in a set containing a single point. We first consider intersections in a single point.

If the cutting plane is not perpendicular to the axis of the cone, then a pair of congruent acute vertical angles is formed by the axis of the cone and its projection in the cutting plane. We define the measure of these acute angles to be the <u>cutting angle</u> of the plane. If the cutting plane is perpendicular to the axis of the cone, we define the cutting angle to be $\frac{\pi}{2}$ in radian measure or 90 in degree measure. If the cutting plane is parallel to the axis of the cone (in this case it may contain the axis), then the cutting angle is defined to be zero. (We could avoid defining these angles in such an unnatural way, were we to consider parallel planes containing the vertex of the cone. However, we are interested solely in the measures of these angles and adopt these definitions.)

<u>Exercises S7-2</u>

1. Prove that any two elements of a right circular cone form congruent acute angles with the axis of the cone.

* 2. Prove that the axis of a right circular cone is the locus of points equidistant from the elements of the cone.

3. Prove that, given any real number except zero as a measure of distance, there exist two distinct points on the axis of a right circular cone which are this measure of distance from the cone.

4. Prove that if a point P on the axis of a right circular cone is at a distance d from the cone, then the locus of points of the cone at a distance d from P is a circle.

S7-3. <u>Spheres of Tangency</u>

Figure 1 is a schematic representation of a plane cutting a cone from a point of view parallel to the cutting plane. V is the vertex of the cone, <u>a</u> is the axis of the cone, ℓ and ℓ' are elements of the cone, α is the elemental angle, β is the cutting angle, P is the point of intersection of the cutting plane and the axis of the cone, and <u>m</u> is the projection of the axis in the cutting plane.

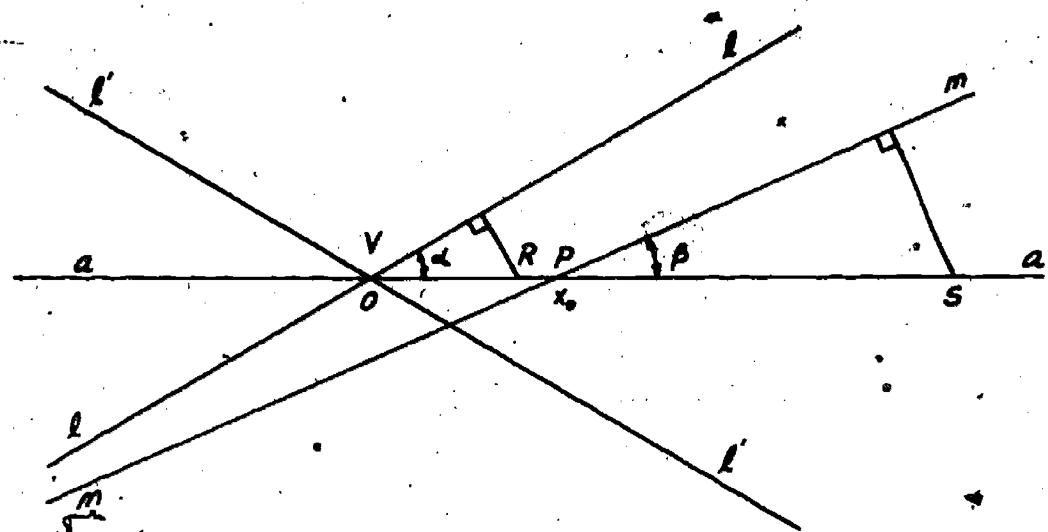


Figure S7-1

We consider three different coordinate systems on line a. In the first coordinate system X the origin is at V; the coordinate of P is positive and is denoted by x_0. The coordinate of an arbitrary point is denoted by x.

The second coordinate system x' is oriented from V to P and assigns to each point R as its coordinate x' the distance from R to the cone, and consequently the radius of the sphere tangent to the cone with center R. This is the case to the right of V. The origin is at V. To the left of V the coordinate is the negative of this radius. This coordinate system is related to the first coordinate system by the following linear equation:

$$x' = x \sin\alpha .$$

The third coordinate system x'' on a is oriented from P to V and assigns to each point S as its coordinate x'' the distance from S to the cutting plane, and consequently the radius of a sphere tangent to the cutting plane with center S. This will be the case to the left of P. The origin is at P. To the right of P the coordinate is the negative of this radius. This coordinate system is related to the first coordinate system by the following linear equation:

$$(x_0 - x)x'' = \sin\beta$$

We observe that, if $x' = x''$, the corresponding point on a is the center of the sphere tangent to the cone and the cutting plane. This is the desired sphere mentioned in Section S7-2.

We equate these two expressions and solve for x :

$$(x_0 - x)x \sin\alpha = \sin\beta$$

$$= x_0 \sin\beta - x \sin\beta$$

$$x \sin\alpha + x \sin\beta = x_0 \sin\beta$$

$$x = x_0 \frac{\sin\beta}{\sin\alpha + \sin\beta}$$

We note that we oriented the first coordinate system in such a way that x_0 was positive and that, inasmuch as α and β are measures of acute angles, $\frac{\sin\beta}{(\sin\alpha + \sin\beta)}$ is between 0 and 1. Hence x is the coordinate of a point between V and P and the radius of the sphere is $x_0\left(\frac{\sin\alpha \sin\beta}{\sin\alpha + \sin\beta}\right)$.

If $\beta > \alpha$, then $\sin\beta > \sin\alpha$, and we discover a second sphere tangent both to the cone and to the cutting plane, but with its center to the <u>right</u> of P. To the right of P the radius of a sphere tangent to the plane is $-x''$. If $x' = -x''$,

$$x \sin\alpha = -(x_0 - x)\sin\beta$$

and

$$x = x_0\left(\frac{\sin\beta}{\sin\alpha - \sin\beta}\right).$$

Since $\frac{\sin\beta}{(\sin\beta - \sin\alpha)} > 1$, x is the coordinate of a point to the right of P. The radius of the sphere is $x_0\left(\frac{\sin\alpha \sin\beta}{\sin\beta - \sin\alpha}\right)$.

If $\beta < \alpha$, then $\sin\beta < \sin\alpha$; we discover a second sphere with center to the left of V. To the left of V the radius of a sphere tangent to the cone is $-x'$. If $-x' = x''$,

$$-x \sin\alpha = (x_0 - x)\sin\beta$$

and

$$x = -x_0\left(\frac{\sin\beta}{\sin\alpha - \sin\beta}\right)$$

where $\frac{\sin\beta}{(\sin\alpha - \sin\beta)} > 1$. Thus x is the coordinate of a point to the left of V, the center of the sphere is more remote from the origin than was that of the first sphere, and the radius is $x_0\left(\frac{\sin\alpha \sin\beta}{\sin\alpha - \sin\beta}\right)$.

If $\beta = \alpha$, $\sin\beta = \sin\alpha$, and the search for other spheres is in vain. The coefficients of x_0 are not defined outside the segment $\overline{VP}$.

Lastly, we consider the possibility that the cutting plane may be parallel to the axis of the cone. In this case the distance from a point on the axis to the plane is constant. Thus $x'' = k$, and following the above argument, we discover that $x = \pm \frac{k}{\sin\alpha}$; there are two spheres, one on either side of V, and each with radius k. We recall that the cutting angle is zero in this case, for the cutting angle is not really the angle itself, but rather a measure associated with the angle.

Degenerate Conic Sections

Before continuing with our discussion of the more elaborate conic sections, we may digress to consider what happens if the cutting plane contains the vertex of the cone. A geometric description should be sufficient. If $\beta > \alpha$, then the vertex is the only point of the section. If $\beta = \alpha$, then the section is a single element of the cone, that is, a line. If $\beta < \alpha$, the section is the union of two elements of the cone, that is, the union of two intersecting lines.

Some sections of the surface called a right circular cylinder are sections of right circular cones. The exceptions are those sections obtained by a cutting plane parallel to the axis of the cylinder, with distance from the axis less than the radius of the cylinder. (The plane may contain the axis.) These sections are the union of two parallel lines. Though not obtainable as sections of cones for algebraic reasons they are included among the degenerate conic sections.

S7-5. Geometric Properties of the Conic Sections

From our consideration of the conic sections so far we may make certain general observations. If $\beta = \frac{\pi}{2}$ (in radians) or 90 (in degrees), it is intuitively obvious and not difficult to prove that this section is a circle. If $\frac{\pi}{2} > \beta > \alpha$, it is apparent that the plane cuts every element of one nappe and that the resulting section is a closed curve. If $\beta = \alpha$, the plane cuts some, but not all of the elements of one nappe. Lastly, if $\beta < \alpha$, the plane cuts some, but not all the elements of each nappe and the curve has two distinct branches.

But to continue our study we need more information. We consider Figure S7-2. We are given a right circular cone with vertex V, axis $\underline{a}$, and elemental angle α. E is a cutting plane, not containing V, with an acute cutting angle β. The conic section is the curve $\underline{s}$. The tangent sphere with center O is tangent to the cone in circle $\underline{c}$ and to the cutting plane at point F.

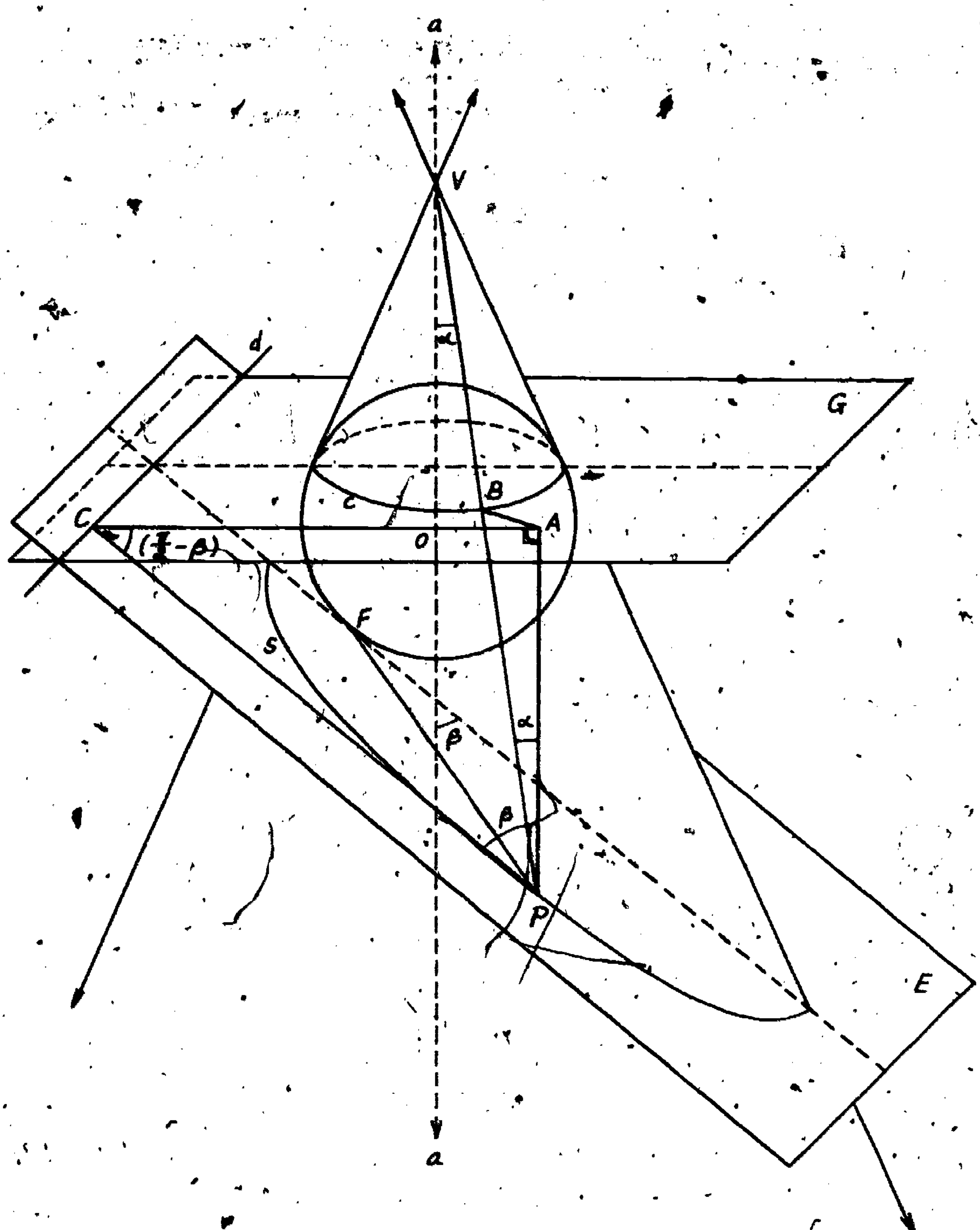


Figure S7-2

Let G be the plane containing circle $\underline{c}$. G is perpendicular to the axis $\underline{a}$, and since E is assumed not to be perpendicular to $\underline{a}$, G and E must form a dihedral angle with edge $\underline{d}$. The plane angle of the dihedral angle is complementary to the cutting angle and has measure $(\frac{\pi}{2} - \beta)$.

Let P be any point of the conic section $\underline{s}$. The plane containing P and perpendicular to $\underline{d}$ intersects the dihedral angle in a plane angle of the dihedral angle which has vertex C and measure $(\frac{\pi}{2} - \beta)$. Let A be the foot of the perpendicular from P to the other side of the plane angle. $\overleftrightarrow{PA}$ is perpendicular to G and $\triangle PAC$ is a right triangle. Since

$m \angle PCA = (\frac{\pi}{2} - \beta)$, $m \angle APC = \beta$ and

$$(1) \qquad \cos \beta = \frac{d(A,P)}{d(P,C)}.$$

We observed that $\overleftrightarrow{AP}$ was perpendicular to G. The axis $\underline{a}$ is also perpendicular to G, so $\underline{a}$ and $\overleftrightarrow{AP}$ are parallel. Consider the element of the cone $\overleftrightarrow{PV}$ which intersects the circle of tangency $\underline{c}$ in point B (which is in G). Since the tangent sphere is between V and the cutting plane, B is between V and P. The elemental angle and $\angle APB$ are a pair of alternate interior angles formed by a transversal of two parallel lines, and consequently $m \angle APB = \alpha$. $\triangle APB$ is a right triangle and

$$(2) \qquad \cos \alpha = \frac{d(A,P)}{d(P,B)}.$$

Both $\overline{PB}$ and $\overline{PF}$ are tangent segments to the sphere from the same point and hence $d(P,F) = d(P,B)$. Substituting in (2), we obtain

$$(3) \qquad \cos \alpha = \frac{d(A,P)}{d(P,F)}.$$

Dividing (1) by (3), we obtain

$$(4) \qquad \frac{\cos \beta}{\cos \alpha} = \frac{d(P,F)}{d(P,C)}.$$

Since both β and α are constant for a given conic section, this quotient is a constant. It is called the <u>eccentricity</u> of the conic section and is denoted by the small letter $\underline{e}$. Geometrically this means that for any point of a given conic section the ratio of its distance from a well-defined point to its distance from a well-defined line is a constant. Both the point,

which is called the focus or focal point, and the line, which is called the directrix, lie in the plane of the conic section. Since we have taken both the elemental angle and the cutting angle to be the measures of acute angles, the eccentricity e will be a positive real number.

We have observed that it is perfectly possible for the cutting plane E to be perpendicular to the axis of the cone. In this case E and G are parallel and the section has no directrix. It does have a focus which is the intersection of the cutting plane and the axis. The section is a circle and the center is at the focus; if U is the focus, then the radius of the circle is $d(U,V) \cdot \tan\alpha$. In this case the expression for the eccentricity would be

$\dfrac{\cos(\frac{\pi}{2})}{\cos\alpha}$, which is zero.

Since this is distinct from the other cases, we may accept it without inconsistency.

We observe that if $\frac{\pi}{2} > \beta > \alpha$, $\cos\beta < \cos\alpha$ and $e < 1$; if $\beta = \alpha$, $\cos\beta = \cos\alpha$ and $e = 1$; if $0 \leq \beta < \alpha$, $1 \geq \cos\beta > \cos\alpha$ and $e > 1$. We take these properties to be definitive for the conic sections.

DEFINITIONS. Given a conic section with eccentricity e:

The conic section is an ellipse if $0 < e < 1$.
The conic section is a parabola if $e = 1$.
The conic section is a hyperbola if $e > 1$.
The conic section is a circle if $e = 0$.

On the other hand, we have shown they may be described by their geometric properties. A circle is the locus of points in a plane at a given distance from a given point, called the center; an ellipse is the locus of points in a plane such that for each point the ratio of its distance from a given point to its distance from a given line is a constant which is less than one; a parabola is the locus of points in a plane such that for each point the ratio of its distance from a given point to its distance from a given line is one; a hyperbola is the locus of points in a plane such that for each point the ratio of its distance from a given point to its distance from a given line is a constant which is greater than one.

Exercises S7-5

1. Prove that if a cutting plane is perpendicular to the axis of a right circular cone, then the sphere of tangency is tangent to the plane at a point on the axis. Prove that in this case the conic section is a circle which centers on the axis.

*2. In Section S7-3 we discovered that if $\beta > \alpha$, there exists a second sphere of tangency such that its center is on the other side of the cutting plane from the vertex. Let this sphere be tangent to the cutting plane at F'. Prove that if P is a point of the section, then $d(P,F) + d(P,F')$ is a fixed constant. In other words, prove that an ellipse is the locus of points in a plane such that for each point, the sum of its distances from two given points in the plane is a fixed constant. (Hint: In Figure S7-2 the second sphere lies below the cutting plane; let $\underline{c}'$ be its circle of tangency. Let B' be the intersection of $\overleftrightarrow{VP}$ and $\underline{c}'$. Then prove that $d(P,F) + d(P,F') = d(B,B')$. Then prove that this distance is the same for all P.)

3. In Section S7-3 we discovered that if $\beta < \alpha$, there exists a second sphere of tangency such that the vertex lies between the centers of the two spheres. Let this sphere be tangent to the cutting plane at F'. Prove that if P is a point of the section, then $|d(P,F) - d(P,F')|$ is a fixed constant. In other words prove that a hyperbola is the locus of points in a plane such that for each point, the absolute value of the difference between its distances from two given points in the plane is a fixed constant. (Hint: In Figure S7-2, the second sphere lies within the upper nappe of the cone; let $\underline{c}'$ be its circle of tangency. Let B' be the intersection of $\overleftrightarrow{VP}$ and $\underline{c}'$. Then prove that $|d(P,F) - d(P,F')| = d(B,B')$. Then prove that this distance is the same for all P.)

*4. Let C be a circle contained in a plane E. The union of the lines perpendicular to E which contain points of C is a <u>right circular cylinder</u>. The lines are called <u>elements</u> of the cylinder; the circle is called a <u>directrix</u> of the cylinder. Prove that the sections of a right circular cylinder are conic sections. Show that in the case of the right circular cylinder there are also spheres of tangency (i.e. tangent to the cylinder in a circle and to the cutting plane at a focal point of the conic section).

In general, the sections of any cone or cylinder, with a conic section as directrix, are also conic sections.

Part 2

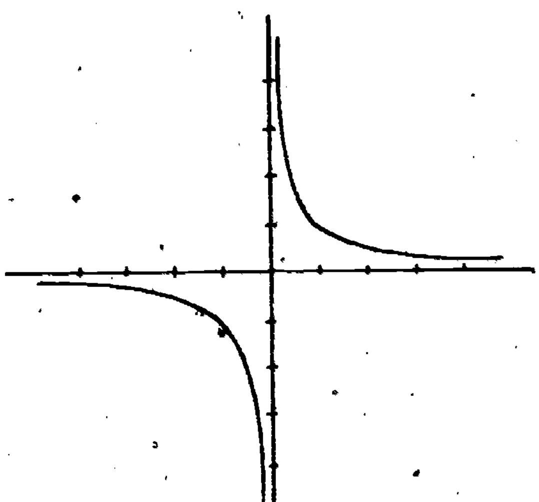
THE GENERAL SECOND-DEGREE EQUATION

S7-6. The General Second-Degree Equation; Rotations and Translations

The conic sections which we have studied have been represented in rectangular coordinates by second-degree equations in two variables. It seems natural to ask whether all equations of second degree in x and y have loci which are conic sections. In its most general form such an equation may be written as

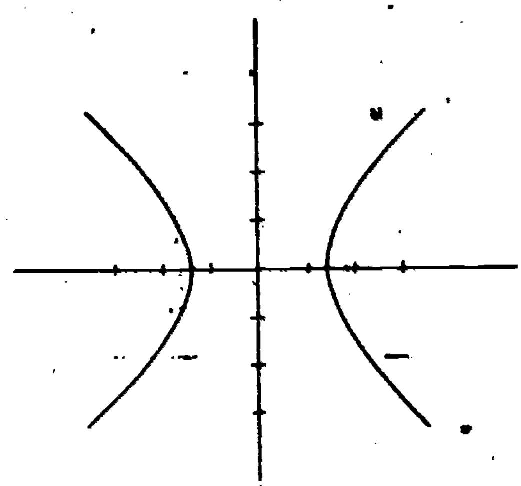
(1) $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A , B , and C are not all zero.

This general form may be difficult to identify, but some techniques which we have used in the preceding sections will permit us to simplify it. The major stumbling block is posed by the xy-term. The only previous equation containing an xy-term, which we have considered in detail, was that of an equilateral hyperbola. We also have another equation for an equilateral hyperbola. Let us consider the graphs of $xy = 1$ and of $\frac{x^2}{2} - \frac{y^2}{2} = 1$.



$xy = 1$

Figure S7-6a



$\frac{x^2}{2} - \frac{y^2}{2} = 1$

Figure S7-6b

The graphs of these two equations seem remarkably similar. Not only are the asymptotes perpendicular in each case, but also the transverse axes are congruent. In fact, it would appear that the graph in Figure S7-6b may be obtained from that in Figure S7-6a by a clockwise rotation of axes through an

angle of 45°. The first equation contains an xy-term, while the second does not. The suggestion is that a rotation of axes as described in Section 4-8 might result in the elimination of the xy-term. It turns out that this is the case, but we are now faced with a second question. What size rotation should we consider? Let us consider the effect of any rotation of axes on the general second-degree equation. We recall that the equations of rotation are:

$$x = x' \cos \theta - y' \sin \theta$$
$$y = x' \sin \theta + y' \cos \theta .$$

If we substitute these values in Equation (1) and expand, we obtain

$$A(x'^2 \cos^2 \theta - 2x'y' \sin \theta \cos \theta + y'^2 \sin^2 \theta)$$
$$+ B(x'^2 \sin \theta \cos \theta - x'y' \sin^2 \theta + x'y' \cos^2 \theta - y'^2 \sin \theta \cos \theta)$$
$$+ C(x'^2 \sin^2 \theta + 2x'y' \sin \theta \cos \theta + y'^2 \cos^2 \theta) + D(x' \cos \theta - y' \sin \theta)$$
$$+ E(x' \sin \theta + y' \cos \theta) + F = 0$$

However, all we want to know is the coefficient of the $x'y'$-term. This is

$$-2A \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) + 2C \sin \theta \cos \theta .$$

If this coefficient is zero, the transformed equation will not contain any $x'y'$-term. If

$$-2A \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) + 2C \sin \theta \cos \theta = 0 ,$$

then

$$B(\cos^2 \theta - \sin^2 \theta) = 2(A - C) \sin \theta \cos \theta .$$

We recall that $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$ and that $2 \sin \theta \cos \theta = \sin 2\theta$. Thus we may write

$$B \cos 2\theta = (A - C) \sin 2\theta ,$$

or, if $A \neq C$,

$$\frac{B}{A - C} = \frac{\sin 2\theta}{\cos 2\theta}$$

or

$$\frac{B}{A - C} = \tan 2\theta .$$

If $A = C$, then

$$B \cos 2\theta = 0$$

or

$$\cos 2\theta = 0 .$$

(We recall that if B were zero, we would not have had to go to all this trouble.) In either case, all we require is a single value of θ which satisfies the appropriate condition. If $\cos 2\theta = 0$, 2θ may be 90°; thus θ may be 45°. If $\tan 2\theta = \frac{B}{A - C}$, which is not zero, we recall that the tangent assumes all non-zero real values once and only once between 0° and 180°. Thus, there exists a unique acute angle θ such that $\tan 2\theta = \frac{B}{A - C}$.

Thus we have shown that in every case in which the second-degree equation has an xy-term, it can be transformed, by a rotation of axes through a unique acute angle, to an equation without an xy-term. The transformed equation has the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$, or, dropping the primes, the form

(2) $Ax^2 + Cy^2 + Dx + Ey + F = 0$. (A and C are never both zero.)

Now the equation is in a form which may be identified more easily. We have already developed techniques for simplifying equations of this form. It is proper to drop the primes only when the <u>form</u> of the equation is being studied.

If AC is not zero, we first complete the squares for the x^2- and x-terms and y^2- and y-terms to obtain

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = \frac{D^2}{4A} + \frac{E^2}{4C} - F, \; AC \neq 0$$

or

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}, \; AC \neq 0 .$$

Now a translation of axes, as introduced in Section 10-2 and described by the equations

$$x = x' - \frac{D}{2A}$$

$$y = y' - \frac{E}{2C},$$

gives the transformed equation

$$Ax'^2 + Cy'^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}, \; AC \neq 0,$$

in which the primes have been omitted for simplicity.

We recognize that if AC is negative and $\frac{CD^2 + AE^2 - 4ACF}{4AC}$ is not zero, or if A, C, and $\frac{CD^2 + AE^2 - 4ACF}{4AC}$ are all positive or all negative, the transformed equation is the equation of a conic section. If A equals C, the conic section is a circle; if AC is positive and A is not equal to C, the conic section is an ellipse; if AC is negative, the conic section is an hyperbola.

We must also consider the case in which $AC = 0$ in Equation (2). Suppose A is zero. Then C is not zero, and we may complete the square for the y^2- and y-terms. Equation (2) is now

$$Cy^2 + Dx + Ey + F = 0,$$

which becomes

$$Cy^2 + Ey = -Dx - F$$

or

$$C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -D\left(x + \frac{F}{D} - \frac{E^2}{4CD}\right)$$

or

$$\left(y + \frac{E}{2C}\right)^2 = -\frac{D}{C}\left(x - \frac{E^2 - 4CF}{4CD}\right).$$

A translation of axes, described by the equations

$$x = x' + \frac{E^2 - 4CF}{4CD},$$

$$y = y' - \frac{E}{2C},$$

gives the transformed equation

$$y^2 = -\frac{D}{C}x.$$

We recognize this as the equation of a parabola, with the vertex at the origin and the axis on the x-axis.

If C is zero, a similar development may be made. The resulting equation will again be of a parabola with the vertex at the origin, but the axis will be on the y-axis.

Exercises §7-6

1. Through what angle must the axes be rotated to eliminate the xy-term from each of the following equations ?

   (a) $x^2 - 4xy + 4y^2 - 4x - 7 = 0$

   (b) $x^2 + \sqrt{3}\,xy + 2y^2 - 3 = 0$

   (c) $x^2 - 3xy + 4y^2 - 9 = 0$

   (d) $x^2 + 3xy - x + y - 1 = 0$

   (e) $3x^2 + 2\sqrt{3}\,xy + y^2 - 2x - 2\sqrt{3} - 16 = 0$

   (f) $12xy + 9y^2 - 2x - 3y - 10 = 0$

2. For each of the following, simplify the equation, identify the conic section, and draw its graph:

   (a) $5x^2 - 6xy + 5y^2 - 8 = 0$

   (b) $5x^2 - 6xy + 5y^2 + 4x + 4y - 4 = 0$

   (c) $7x^2 + 2\sqrt{3}\,xy + 5y^2 - 16 = 0$

   (d) $3x^2 + 2xy + 3y^2 + 4x + 4y = 0$

   (e) $x^2 - 6xy + y^2 + 14x + 10y + 14 = 0$

   (f) $11x^2 + 24xy + 4y^2 - 44x - 48y + 24 = 0$

   (g) $2xy + 4x - 4y - 9 = 0$

   (h) $9x^2 - 24xy + 16y^2 + 90x + 130y = 0$

This treatment of the quadratic equations which describe conic sections has been solely concerned with techniques employed in simplifying the equations. It is important that we also consider what we have done from a geometric point of view.

In Section 6-2 we have stressed the importance of recognizing symmetries in figures, both as an aid in the sketching of graphs of equations and as a guide in the selection and orientation of a coördinate system to describe a graph by an equation. In particular we have considered axes of symmetry and points of symmetry. We have observed that in rectangular coordinates the y-axis is an axis of symmetry for a locus described by $f(x,y) = 0$ if $f(x,y) = f(-x,y)$ and that the x-axis is an axis of symmetry if $f(x,y) = f(x,-y)$. The origin is a point of symmetry if $f(x,y) = f(-x,-y)$.

The origin is always a point of symmetry if both the x-axis and the y-axis are axes of symmetry. However, the converse of this last statement is not true. (Consider $y = x^3$.)

It was in Section 10-2 that we first overtly considered translations of axes as a means to simplify the analysis of the graph of an equation. However, we have really used this technique before. Do you recall that in Chapter 2 in our discussion of direction angles and direction cosines for a line we found it convenient to consider a parallel line through the origin?

In our rather mechanical treatment of quadratic equations in this section we have been guided by symmetries in the graphs of the equations. The rotations of axes which we performed in Section 10-3 made an axis of symmetry parallel to a coordinate axis. The translations of axes made a point of symmetry also be the origin. (In the case of the parabola there is no point of symmetry. The translation of axes made the vertex be the origin as well.)

It is possible to describe points and axes of symmetry quite generally.

<u>DEFINITIONS</u>. Let S be a set of points. The segments joining points of S are <u>chords</u> of the set. If there exists a point P such that, for each point X of S, the segment with end-point X and mid-point P is a chord of the set, then P is a <u>point of symmetry</u> or <u>center</u> of S.

Let S be a set of points in a plane and let L be a line in the plane. If, for every point X of S, the segment which

(i) has end-point X,

(ii) is perpendicular to L,

and (iii) has its mid-point on L,

is a chord of S, then L is an <u>axis of symmetry</u> of S.

S7-7. The General Second-Degree Equation, Translation and Rotation

In simplifying second-degree equations, it is in some cases more convenient to translate the axes first to eliminate the x- and y-terms. Then we rotate the new axes to eliminate the xy-term.

If we start again with Equation (1) of Section S7-6 and use the equations of translation

$$x = x' + h$$
$$y = y' + k ,$$

we obtain

$$A(x'^2 + 2hx' + h^2) + B(x'y' + kx' + hy' + hk) + C(y'^2 + 2ky' + k^2)$$
$$+ D(x' + h) + E(y' + k) + F = 0 .$$

If we collect terms, this becomes

$$(1)\quad Ax'^2 + Bx'y' + Cy'^2 + (2Ah + Bk + D)x' + (Bh + 2Ck + E)y'$$
$$+ (Ah^2 + Bhk + Ck^2 + Dh + Ek + F) = 0 .$$

We note that the coefficients of the second-degree terms will not be changed by a translation of axes. If we can find values of h and k such that

$$2Ah + Bk + D = 0$$

and

$$Bh + 2Ck + E = 0 ,$$

we shall be able to substitute these values in Equation (1) to obtain a transformed equation free of first-degree terms. We can solve this pair of equations to obtain

$$h = \frac{\begin{vmatrix} -D & B \\ -E & 2C \end{vmatrix}}{\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix}} \quad \text{and} \quad k = \frac{\begin{vmatrix} 2A & -D \\ B & -E \end{vmatrix}}{\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix}}$$

if

$$\delta = \begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix} = 4AC - B^2 \neq 0 .$$

The determinant δ is of some interest in the analysis of the second-degree equation and is sometimes called the characteristic.

You should sense that, when it is possible, it is easier to translate the axes first and then perform a rotation of the new axes. The fewer terms there are in an equation, the easier it is to perform a rotation. However, if the characteristic is zero, we cannot find the appropriate values of h and k. We have no choice but to follow the procedure of Section 6-8.

If the characteristic is not zero, the transformed equation is

$$Ax'^2 + Bx'y' + Cy'^2 + F' = 0$$

where

$$F' = Ah^2 + Bhk + Ck^2 + Dh + Ek + F .$$

It is easy to remember what F' is if you notice that when we represent the original equation by $f(x,y) = 0$, then $F' = f(h,k)$.

Exercises S7-7a

1. Find h and k such that a translation of axes described by

$$x' = x + h$$
$$y' = y + k$$

will eliminate the first-degree terms of

$$4x^2 + y^2 - 8x + 4y + 4 = 0 .$$

Verify for this case that the constant term in the transformed equation is equal to $f(h,k)$.

2. Transform each of the following equations by first translating the axes so as to eliminate the first-degree terms. Then rotate the axes to remove the xy-term. Sketch the curve, showing old and new axes.

(a) $8x^2 - 4xy + 5y^2 - 24x + 24y = 0$

(b) $3x^2 + 10xy + 3y^2 - 4x + 22y - 53 = 0$

(c) $7x^2 - 24xy + 120x + 144 = 0$

(d) $4x^2 - 8xy + 4y^2 - 9\sqrt{2}\,x + 7\sqrt{2}\,y + 14 = 0$

Once again it's important that we consider this method of simplifying the second-degree from a geometric point of view. Why can't we find an appropriate translation of axes when the characteristic is zero? You should recall that in the previous section we observed that the translation of axes makes the new origin a point of symmetry. Our search for values of h and k is in fact a search for the coordinates of a point of symmetry. Since the parabola has no point of symmetry, the characteristic of its equation turns out to be zero. The converse of this statement is not necessarily true, but we shall defer the consideration of this question.

If we approach the analysis of the second-degree equation from a geometric point of view, we can develop methods which may be applied to more complicated problems.

First we observe that if a set of points in a plane has an axis of symmetry, then the axis of symmetry is the perpendicular bisector of chords joining pairs of points of the set. In fact, every point of the set is an endpoint of such a chord. We have already noted that the equation of a locus is frequently simplified if an axis of symmetry of the locus is parallel to one of the coordinate axes. We shall first find an axis of symmetry for the graph of the second-degree equation and then rotate the axes to make one of them parallel to this axis of symmetry. Since the chords in the definition of an axis of symmetry are all perpendicular to the axis of symmetry, they are parallel to each other. Then the lines determined by the chords have parametric representations in terms of a fixed pair of direction cosines (λ,μ). Let (x',y') be the midpoint of a chord. Then the parametric representation of the line containing the chord is

$$x = x' + \lambda t$$
$$y = y' + \mu t .$$

When (x,y) is an endpoint of the chord, the coordinates should satisfy the second-degree equation. If we substitute the parametric representation of the endpoint in the second-degree equation, we obtain

$$A(x'^2 + 2\lambda tx' + \lambda^2t^2) + B(x'y' + \mu tx' + \lambda ty' + \lambda\mu t^2)$$
$$+ C(y'^2 + 2\mu ty' + \mu^2t^2) + D(x' + t) + E(y' + \mu t) + F = 0 .$$

If we collect terms in t^2, λt and μt, we obtain

(2) $$(A\lambda^2 + B\lambda\mu + C\mu^2)t^2 + (2Ax' + By' + D)\lambda t + (Bx' + 2Cy' + E)\mu t$$
$$+ (Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F) = 0 .$$

Now we observe that both endpoints of the chord must satisfy the equation. Furthermore, if t_1 is the value of the parameter at one endpoint, $-t_1$ is the value of the parameter at the other endpoint. This must be the case for any chord and any equation. This implies that the form of the equation in t must always be

$$t^2 - t_1^2 = 0 .$$

Thus in Equation (2) the coefficient of t, or

(3) $$(2Ax' + By' + D)\lambda + (Bx' + 2Cy' + E)\mu ,$$

must be zero. Now λ and μ are fixed for any particular second-degree equation, but x' and y' are variables, designating the coordinates of the midpoints of the chords perpendicular to the axis of symmetry. But the midpoints of the chords are on the axis of symmetry. Thus the condition on Expression (3) written as a linear equation in x' and y' is the equation of the axis of symmetry:

(4) $$(2A\lambda + B\mu)x' + (B\lambda + 2C\mu)y' + (D\lambda + E\mu) = 0 .$$

This equation is in the general form. Hence, $(2A\lambda + B\mu, B\lambda + 2C\mu)$ is a pair of direction numbers for normals to the axis of symmetry. But so is (λ, μ). Therefore, for some non-zero real number k

(5) $$2A\lambda + B\mu = k\lambda$$
and
$$B\lambda + 2C\mu = k\mu .$$

If we solve the second equation for μ, we obtain

$$\mu = \frac{-B}{2C - k} \cdot \lambda .$$

We substitute in the first equation, which becomes

$$(2A - k)\lambda - \frac{B^2}{2C - k}\lambda = 0$$

or
$$(4AC - 2Ak - 2Ck + k^2)\lambda - B^2 = 0$$

or
$$[k^2 - 2(A + C)k + (4AC - B^2)]\lambda = 0 .$$

Now either λ or the coefficient must be zero. But if λ were zero, μ would also be zero, which is impossible, since (λ, μ) is a pair of direction cosines and $\lambda^2 + \mu^2 = 1$. Therefore,

(6) $$k^2 - 2(A + C)k + (4AC - B^2) = 0 .$$

Equation (6) is called the <u>characteristic equation</u> for the given second-degree equation and its roots are called <u>characteristic values</u> for the quadratic equation. We note that the sum of the roots is $2(A + C)$ while the product of the roots is $4AC - B^2$ or δ, the characteristic of the quadratic equation.

We may then solve Equation (6) for k and substitute these values in Equations (5) to determine the pairs of direction cosines (λ, μ). These pairs of values may then be substituted in (4) to obtain the equations of axes of symmetry. We note that if the characteristic is zero, Equation (6) has only one non-zero root. In Equation (5) k must be non-zero; hence, only one pair of direction cosines may be obtained, and the graph of the quadratic equation will have only one axis of symmetry. This is consistent with our previous observations that the parabola has only one axis of symmetry and that the characteristic of its equation is zero. We also note that the characteristic equation will have equal roots only if

$$(A + C)^2 = 4AC - B^2$$

or

$$A^2 + 2AC + C^2 = 4AC - B^2$$

or

$$A^2 - 2AC + C^2 = -B^2$$

or

$$(A - C)^2 = -B^2 .$$

This may only be true if B is zero and A equals C. When this is the case, you will recall that the graph of the quadratic equation is a circle. Equations (5) are satisfied by any pair of direction cosines, and there are infinitely many equations (4). This is not surprising inasmuch as every diameter of a circle determines an axis of symmetry. It is a fact that the characteristic equation of a quadratic equation always has real roots. Furthermore, if these roots determine two axes of symmetry, these axes are perpendicular. We are familiar with the fact that the intersection of two perpendicular axes of symmetry is a point of symmetry. This suggests one way to find a point of symmetry.

We may also discover points of symmetry from the definition of point of symmetry given in Section 6-8 and from the conditions on Expression (3) above:

(7) $$(2Ax' + By' + D)\lambda + (Bx' + 2Cy' + E)\mu = 0 .$$

You should recall that (x', y') is the midpoint of a chord of the graph while (λ, μ) is a pair of direction cosines in the parametric representation of the chord. When we wanted to find an axis of symmetry, λ and μ were fixed while (x', y') was variable. However, here we want to find a fixed point (x', y') which will satisfy Equation (7) for all pairs (λ, μ). This will be the case only if the coefficients of λ and μ are both zero; that is, if

(8) $$2Ax' + By' + D = 0$$

and $$Bx' + 2Cy' + E = 0 .$$

A solution of this pair of equations will be a point of symmetry or center of the graph of the second-degree equation. The pair of equations will have a unique solution if

$$\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix} = 4AC - B^2 = \delta \neq 0 .$$

Example 1. Find the axes of symmetry and center of the graph of $8x^2 - 4xy + 5y^2 - 36x + 18y + 9 = 0$.

Solution. The characteristic equation [Equation (6)] becomes

$$k^2 - 2(8 + 5) + 4(8)(5) - (4)^2 = 0$$

or

$$k^2 - 26k + 144 = 0$$

or

$$(k - 8)(k - 18) = 0 .$$

The characteristic values are 8 and 18. Now Equations (5) become

$$2(8)\lambda + (-4)\mu = 8\lambda \qquad\qquad 2(8)\lambda + (-4)\mu = 18\lambda$$
and
$$(-4)\lambda + 2(5)\mu = 8\mu \qquad\qquad (-4)\lambda + 2(5)\mu = 18\mu$$

or

$$8\lambda - 4\mu = 0 \qquad\qquad 2\lambda + 4\mu = 0$$
and
$$-4\lambda + 2\mu = 0 \qquad\qquad 4\lambda + 8\mu = 0 .$$

These pairs of equations are dependent, but since $\lambda^2 + \mu^2 = 0$, we may obtain the solutions $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

If we substitute these values in Equation (4), we obtain the equations of the axes of symmetry:

$$[2(8)\frac{1}{\sqrt{5}} + (-4)\frac{2}{\sqrt{5}}]x + [(-4)\frac{1}{\sqrt{5}} + 2(5)\frac{2}{\sqrt{5}}]y + [(-36)\frac{1}{\sqrt{5}} + 18\frac{2}{\sqrt{5}}] = 0$$

or $8x + 16y = 0$

or $x + 2y = 0$,

and

$$[2(8)(\frac{-2}{\sqrt{5}}) + (-4)\frac{1}{\sqrt{5}}]x + [(-4)(\frac{-2}{\sqrt{5}}) + 2(5)\frac{1}{\sqrt{5}}]y + (-36)(\frac{-2}{\sqrt{5}}) + 18\cdot\frac{1}{\sqrt{5}}] = 0$$

or $-36x + 18y + 90 = 0$

or $2x - y - 5 = 0$.

Equations (8) will enable us to find the center. The pair of equations

$$2(8)x + (-4)y + (-36) = 0$$
$$(-4)x + 2(5)y + 18 = 0$$

or

$$4x - y = 9$$
$$-4x + 10y = -18$$

has the unique solution $(2,-1)$. The point is the center or point of symmetry for the graph. We note that this point is also the intersection of the axes of symmetry.

Exercises S7-7b

Find the axes of symmetry and centers, if any, of the graphs:

1. $xy + 5x - 2y - 10 = 0$
2. $2x^2 + xy - 6y^2 + 7x - 7y + 3 = 0$

S7-8. Degenerate and Imaginary Conics and the Discriminant Δ

In our treatment of the second-degree or quadratic equation in the previous two sections, we have restricted our discussion to equations with graphs which are proper conic sections. We have made certain restrictions on the constants of the equation. In this section we shall relax these restrictions and consider the loci, if any, of the resulting equations. We shall also develop means of identifying and classifying the various possibilities. We have already encountered the degenerate conic sections whose graphs are single points, or pairs of lines which may be parallel, concurrent, or coincident. We have also considered equations whose loci are empty, but which are called imaginary circles and imaginary ellipses because of the form of their equations.

In Section 6-3 we have considered the problem of factoring functions. If we can factor the left member of the equation,

(1) $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, and C are not all zero,

into two linear factors, we would know that the graph is the union of two lines. Under what conditions is this expression factorable? You should recall that quadratic equations in a single variable often may be solved by

factoring the quadratic expression into linear factors. Such an equation may always be solved by completing the square or by using the quadratic formula, which is equivalent to completing the square. In all likelihood on some occasion you have failed to detect the linear factors in the quadratic member of an equation and have resorted to the quadratic formula, only to discover that the equation really could have been solved by factoring. This suggests that the quadratic formula may be an aid in finding linear factors. In fact, the quadratic expression $ax^2 + bx + c$ may always be expressed as the product of linear factors as

$$ax^2 + bx + c = a\left(x - \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\right)\left(x - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right)$$

if we allow the use of complex numbers when necessary.

Now Equation (1) may be considered to be a quadratic equation in x if A is not zero, or in y if C is not zero. Let us assume that C is not zero and write Equation (1) as

(2) $$Cy^2 + (Bx + E)y + (Ax^2 + Dx + F) = 0, \quad C \neq 0.$$

Then

(3) $$y = \frac{-(Bx + E) \pm \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C}.$$

The discriminant involves terms in x^2 and x, but if it is a perfect square, we may eliminate the radical to obtain two expressions for y, say α and β, which are linear in x (i.e. α and β involve only x to the first power and various constants). Then Equation (2) and, if C is not zero, Equation (1) may be written as

(4) $$C(y - \alpha)(y - \beta) = 0,$$

where the factors of the left member are linear in x and y. The graph of Equation (4), and consequently of Equation (2), is the union of the graphs of

$$y - \alpha = 0$$
$$y - \beta = 0,$$

which are lines. However, the conclusion of this argument does not hold unless the discriminant of Equation (2) is a perfect square. The discriminant is

$$(Bx + E)^2 - 4C(Ax^2 + Dx + F),$$

as seen in Expression (3), or

(5) $$(B^2 - 4AC)x^2 + 2(BE - 2CD)x + (E^2 - 4CF) .$$

Again we make use of the quadratic formula as an aid in factoring. Expression (5) will be a perfect square if and only if the roots of the equation

(6) $$(B^2 - 4AC)x^2 + 2(BE - 2CD)x + (E^2 - 4CF) = 0$$

are equal. These roots will be equal if and only if the discriminant of Equation (6) is zero. This discriminant is

$$4(BE - 2CD)^2 - 4(B^2 - 4AC)(E^2 - 4CF) ,$$

which will be zero if and only if

$$B^2E^2 - 4BCDE + 4C^2D^2 - B^2E^2 + 4B^2CF + 4ACE^2 - 16AC^2F = 0$$

or $$-2C(2BDE - 2CD^2 - 2B^2F - 2AE^2 + 8ACF) = 0$$

or (7) $$8ACF - 2AE^2 - 2B^2F + BDE + BDE - 2CD^2 = 0$$

or $$2A(4CF - E^2) - B(2BF - DE) + D(BE - 2CD) = 0$$

or $$2A\begin{vmatrix} 2C & E \\ E & 2F \end{vmatrix} - B\begin{vmatrix} B & D \\ E & 2F \end{vmatrix} + D\begin{vmatrix} B & D \\ 2C & E \end{vmatrix} = 0 ,$$

or $$\begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = \Delta = 0 .$$

This determinant Δ is called the <u>discriminant</u> of the second-degree equation. If Δ is zero, the roots of Equation (6) are equal and the Expression (5), which is the discriminant in Equation (2), is a perfect square. Thus the graph of Equation (2) is the union of two lines; if C is not zero, this set is also the graph of Equation (1).

If C is zero and A is not zero, we could go through a similar argument, treating the second-degree equation as a quadratic equation in x. Eventually we should discover that if Equation (7) holds and A is not zero, then the graph of Equation (1) is the union of two lines. But Equation (7) is equivalent to $\Delta = 0$.

If both A and C are zero, then B cannot be zero (or else the equation would no longer be of second degree), and Equation (1) reduces to

$$Bxy + Dx + Ey + F = 0 , \quad B \neq 0 .$$

The graph will be the union of two lines if

$$Bxy + Dx + Ey + F$$

may be expressed as the product of linear factors, or as $B(x + a)(y + b)$. Now

$$Bxy + Dx + Ey + F = B(x + a)(y + b) \quad \text{for all } x \text{ and } y$$

or

$$Bxy + Dx + Ey + F = Bxy + Bbx + Bay + Bab \quad \text{for all } x \text{ and } y$$

if and only if $D = Bb$, $E = Ba$, and $F = Bab$. In this case $DE = BF$ or $BF - DE = 0$.

If A, C, and $BF - DE$ are all zero, then

$$\Delta = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = \begin{vmatrix} 0 & B & D \\ B & 0 & E \\ D & E & 2F \end{vmatrix}.$$

$$= -B \begin{vmatrix} B & D \\ E & 2F \end{vmatrix} + D \begin{vmatrix} B & D \\ 0 & E \end{vmatrix} = -B(2BF - DE) + D(BE)$$

$$= -2B^2F + BDE + BDE = -2B(BF - DE) = 0 .$$

In summary, if the graph of a second-degree equation is the union of two lines, then the discriminant is zero. The arguments which we have developed are reversible, although we have not attempted to show this here. Hence, the converse of the above is also true. If the discriminant of the general second-degree equation is zero, the left member of the equation may be expressed as the product of linear factors.

We have not considered carefully what lines, if any, these factors might represent. If Expression (5) is a perfect square, the factors are linear, but suppose that $B^2 - 4AC$, the coefficient of x^2, is negative? We note that this is the condition when the characteristic δ is positive. In this case the coefficients in the square root are complex numbers, as are the coefficients in the linear factors. What sort of "lines" could these factors possibly represent? We shall not attempt to explore this question in detail. It is sufficient for our needs to observe that even though the coefficients are complex numbers, there still are real values which satisfy the corresponding equations. For example, the pair of equations

$$y + (2 + i)x - 1 = 0$$
$$y - (4 - 2i)x + 6 - 2i = 0$$

has the solution $(1,-2)$. This is always the case for the linear factors which we encounter here. The value of x which satisfies Equation (6) is real, as is the corresponding value of y. These real values are the coordinates of the point of intersection of the graphs of the corresponding linear equations. Thus, when the discriminant is zero and the characteristic is positive, the locus of a quadratic equation is a point. It is not possible that the linear factors represent dependent or inconsistent equations, for the coefficients of x and y cannot be proportional. (Why?)

If both the discriminant and the characteristic are zero, Expression (5) is a perfect square only if it reduces to $E^2 - 4CF$. (Why?) The locus of the equation will be empty, two coincident lines, or two parallel lines according as $E^2 - 4CF$ is negative, zero, or positive.

If the discriminant is zero and the characteristic is negative, we note that $E^2 - 4CF$ must be non-negative. Otherwise, Expression (5) would only be a perfect square if the coefficient of x were complex, which is impossible. The linear factors cannot represent dependent or inconsistent equations (Why?), and the locus of the second-degree equation is two intersecting lines.

Example. Find the locus of $2x^2 + xy - 6y^2 + 7x - 7y + 3 = 0$.

Solution. We determine that $\Delta = 0$, and seek to factor the left member of the equation by grouping the second-degree terms.

$$2x^2 + xy - 6y^2 + 7x - 7y + 3$$
$$= (2x - 3y)(x + 2y) + (7x - 7y) + 3 .$$

By inspection and trial we discover the factors

$$(2x - 3y + 1)(x + 2y + 3) .$$

Hence the quadratic equation may be written

$$(2x - 3y + 1)(x + 2y + 3) .$$

The locus of the equation is two intersecting lines. If we had not been able to find factors in this way, we could have considered the equation to be a quadratic equation in one variable, say y as above, and could have used the quadratic formula to determine the factors.

Exercises S7-8

1. Determine whether the following equations represent degenerate conic sections. If so, find the linear factors of the left member and the graph.

   (a) $6xy + 3x - 8y - 4 = 0$

   (b) $2x^2 + 8xy - x + 4y - 1 = 0$

   (c) $4x^2 - 5xy + 9y^2 - 1 = 0$

   (d) $2x^2 - xy - 6y^2 = 0$

2. If the discriminant of a second-degree equation is zero, but the characteristic is not zero, why cannot the linear factors of the left member of the equation represent dependent or inconsistent linear equations?

3. If both the discriminant and the characteristic of a quadratic equation are zero, show why Expression (5) must reduce to $E^2 - 4CF$. Why must the linear factors represent dependent or inconsistent equations?

S7-9. Invariants of the Second-Degree Equation

We have made many observations and devised several tests for the second-degree equation. We have obtained these results with the equation written in special forms. We shall show that the values of the characteristic δ and the discriminant Δ, as well as certain other algebraic expressions, are not changed by the transformations which we have used. We shall say that these values are invariant under translation and rotation of axes.

We consider a translation of axes as described in Section S7-7. If we denote the new coefficients by primes, we have

$$\begin{aligned}
A' &= A \\
B' &= B \\
C' &= C \\
D' &= 2Ah + Bk + D \\
E' &= Bh + 2Ck + E \\
F' &= Ah^2 + Bhk + Ck^2 + Dh + Ek + F.
\end{aligned}$$

We note that A, B, C, $A + C$, and consequently δ are invariant. To show that the discriminant is unchanged we consider

$$\Delta' = \begin{vmatrix} 2A' & B' & D' \\ B' & 2C' & E' \\ D' & E' & 2F' \end{vmatrix} = \begin{vmatrix} 2A & B & 2Ah + Bk + D \\ B & 2C & Bh + 2Ck + E \\ 2Ah + Bk + D & Bh + 2Ck + E & 2(Ah^2 + Bhk + Ck^2 + Dh + Ek + F) \end{vmatrix}.$$

We recall that adding a linear combination of several rows or columns to yet another row or column does not change the value of the determinant. We first try to make the upper right element be D. We multiply the elements of the first column by $-h$, those of the second column by $-k$, and add the sum to the third column to obtain

$$\Delta' = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ 2Ah + Bk + D & Bh + 2Ck + E & Dh + Ek + 2F \end{vmatrix}.$$

To make the lower left element be D, we multiply the elements of the first row by $-h$, those of the second row by $-k$, and add the sum to the third row. Thus

$$\Delta' = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = \Delta,$$

and we have shown the discriminant to be invariant under translation of axes.

Now we consider a rotation of axes as described in Section S7-6. If we denote the new coefficients by primes, we have

$$\begin{aligned}
A' &= A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta \\
B' &= -2A\sin\theta\cos\theta + B\cos^2\theta - B\sin^2\theta + 2C\sin\theta\cos\theta \\
&= B(\cos^2\theta - \sin^2\theta) - 2(A - C)\sin\theta\cos\theta \\
&= B\cos 2\theta - (A - C)\sin 2\theta \\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\
D' &= D\cos\theta + E\sin\theta \\
E' &= -D\sin\theta + E\cos\theta \\
F' &= F.
\end{aligned}$$

In this case the coefficients in δ' and Δ' become quite complicated. We will first consider certain simpler expressions involving the coefficients. We shall then use these results to prove that δ and Δ are invariant.

We note that F is invariant. $A + C$ is also invariant, for

$$A' + C' = A(\cos^2 \theta + \sin^2 \theta) + C(\sin^2 \theta + \cos^2 \theta)$$
$$= A + C .$$

$(A - C)^2 + B^2$ is invariant, for

$$A' - C' = (A - C)\cos^2 \theta + 2B \sin \theta \cos \theta + (C - A)\sin^2 \theta$$
$$= (A - C)(\cos^2 \theta - \sin^2 \theta) + B(2 \sin \theta \cos \theta)$$
$$= (A - C)\cos 2\theta + B \sin 2\theta$$

and

$$(A' - C')^2 + B'^2 = (A - C)^2\cos^2 2\theta + 2B(A - C)\cos 2\theta \sin 2\theta + B^2 \sin^2 2\theta$$
$$+ B^2 \cos^2 2\theta - 2B(A - C)\cos 2\theta \sin 2\theta + (A - C)^2 \sin^2 2\theta$$
$$= (A - C)^2(\cos^2 2\theta + \sin^2 2\theta) + B^2(\sin^2 2\theta + \cos^2 2\theta)$$
$$= (A - C)^2 + B^2 .$$

Also $D^2 + E^2$ is invariant, for

$$D'^2 + E'^2 = D^2 \cos^2 \theta + 2DE \cos \theta \sin \theta + E^2 \sin^2 \theta$$
$$+ D^2 \sin^2 \theta - 2DE \cos \theta \sin \theta + E^2 \cos^2 \theta$$
$$= D^2(\cos^2 \theta + \sin^2 \theta) + E^2(\sin^2 \theta + \cos^2 \theta)$$
$$= D^2 + E^2 .$$

Now

$$\delta = 4AC - B^2$$
$$= (A + C)^2 - (A - C)^2 - B^2$$
$$= (A + C)^2 - [(A - C)^2 + B^2] .$$

Since $(A + C)^2$ and $(A - C)^2 + B^2$ are invariant, their difference, which is the characteristic, is invariant under rotation of axes.

It remains to show that the discriminant Δ is invariant under rotation. We recall from Section S7-8, Equation (7) that

$$\Delta = 8ACF - 2AE^2 - 2B^2F + 2BDE - 2CD^2 .$$

We rewrite this as

$$\Delta = 8ACF - 2B^2F + 2BDE - (AE^2 + AD^2 + CE^2 + CD^2) - (AE^2 - AD^2 - CE^2 + CD^2)$$

or $\Delta = 2F(4AC - B^2) + 2BDE - (A + C)(E^2 + D^2) - (A - C)(E^2 - D^2)$.

We have already noted that F, $4AC - B^2$, $A + C$, and $E^2 + D^2$ are invariant. Thus, the first and third terms are invariant. We still must show that $2BDE - (A - C)(E^2 - D^2)$ is invariant.

Now $2B'D'E' = 2[B \cos 2\theta - (A - C)\sin 2\theta](D \cos\theta + E \sin\theta)(-D \sin\theta + E \cos\theta)$

$= 2[B \cos 2\theta - (A - C)\sin 2\theta][-D^2 \sin\theta \cos\theta + E^2 \sin\theta \cos\theta + DE(\cos^2\theta - \sin^2\theta)]$

$= [B \cos 2\theta - (A - C)\sin 2\theta][(E^2 - D^2)(2 \sin\theta \cos\theta) + 2DE(\cos^2\theta - \sin^2\theta)]$

$= [B \cos 2\theta - (A - C)\sin 2\theta][(E^2 - D^2)\sin 2\theta + 2DE \cos 2\theta]$,

$E'^2 - D'^2 = (-D \sin\theta + E \cos\theta)^2 - (D \cos\theta + E \sin\theta)^2$

$= D^2 \sin^2\theta - 2DE \sin\theta \cos\theta + E^2 \cos^2\theta - D^2 \cos^2\theta - 2DE \sin\theta \cos\theta - E^2 \sin^2\theta$

$= (E^2 - D^2)(\cos^2\theta - \sin^2\theta) - 2DE(2 \sin\theta \cos\theta)$

$= (E^2 - D^2)\cos 2\theta - 2DE \sin 2\theta$,

and

$$A' - C' = (A - C)\cos 2\theta + B \sin 2\theta .$$

Thus,

$2B'D'E' - (A' - C')(E'^2 - D'^2) = [B \cos 2\theta - (A - C)\sin 2\theta][(E^2 - D^2)\sin 2\theta + 2DE \cos 2\theta]$

$\qquad - [B \sin 2\theta + (A - C)\cos 2\theta][(E^2 - D^2)\cos 2\theta - 2DE \sin 2\theta]$

$\quad = \cos^2 2\theta[2BDE - (A - C)(E^2 - D^2)]$

$\qquad + \sin^2 2\,[-(A - C)(E^2 - D^2) + 2BDE]$

$\qquad + \sin 2\theta \cdot \cos 2\theta[B(E^2 - D^2) - (A - C)(2DE) - B(E^2 - D^2) + (A - C)(2DE)]$

$\quad = (\sin^2 2\theta + \cos^2 2\theta)[2BDE - (A - C)(E^2 - D^2)]$

$\quad = 2BDE - (A - C)(E^2 - D^2)$.

Thus the discriminant of the second-degree equation is also invariant under rotation.

We note that if the graph of the second-degree equation has a point of symmetry, or represents a <u>central conic</u>, then after a translation of the axes which makes the new origin the point of symmetry, the new equation is

$$A'x^2 + B'xy + C' + F' = 0 ,$$

for which

$$\Delta' = \begin{vmatrix} 2A' & B & 0 \\ B & 2C' & 0 \\ 0 & 0 & \text{[illegible]} \end{vmatrix}$$

$$= 2F' \begin{vmatrix} 2A' & B \\ B & 2C' \end{vmatrix}$$

$$= 2F'\delta'$$

$$F' = \frac{\Delta'}{2\delta'} ,$$

but since Δ and δ are invariant under translation,

$$F' = \frac{\Delta}{2\delta} ;$$

and the transformed equation is

$$Ax^2 + Bxy + Cy^2 + \frac{\Delta}{2\delta} = 0 .$$

S7-10. Summary

We have shown that if the locus of a second-degree equation is not empty, then the graph is either a proper conic section or a degenerate conic section. We have developed many methods and criteria for analyzing such equations and have found certain invariants called the characteristic and discriminant particularly important. We summarize some of these results in the form of a table.

	$\delta < 0$	$\delta = 0$	$\delta > 0$
$\Delta = 0$	intersecting lines	empty, or parallel or coincident lines	point-ellipse or point-circle.
$\Delta \neq 0$	hyperbola	parabola	circle, ellipse, or empty

Example. Discuss the locus of

$$8x^2 - 4xy + 5y^2 - 36x + 18y + 9 = 0 .$$

Solution. Here $\Delta = -10,368$ and $\delta = 144$.

Since $B \neq 0$, the locus may not be a circle, but may be an ellipse.

$$F' = \frac{\Delta}{2\delta} = -36 ,$$

so the locus is a real ellipse.

If we substitute coefficients in the equations

$$2Ah + Bk + D = 0$$
$$Bh + 2Ck + E = 0$$

we obtain

$$16h - 4k - 36 = 0$$
$$-4h + 10k + 18 = 0 \,,$$

which give $(2,-1)$ as the center of the ellipse. The characteristic equation

$$k^2 - 2(A + C)k + (4AC - B^2) = 0$$

is

$$k^2 - 26k + 144 = 0 \,,$$

which gives 8 and 18 as the characteristic values. These are substituted in the equations

$$2A\lambda + B\mu = k\lambda$$
$$B\lambda + 2C\mu = k\mu$$

to obtain

$$8\lambda - 4\mu = 0 \quad \text{and} \quad 2\lambda + 4\mu = 0$$
$$-4\lambda + 2\mu = 0 \qquad\qquad 4\lambda + 8\mu = 0 \,,$$

which give $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ as pairs of direction cosines for the axes of symmetry

$$(2A\lambda + B\mu)x + (B\lambda + 2C\mu)y + (D\lambda + E\mu) = 0$$

or

$$x + 2y = 0$$
$$2x - y - 5 = 0 \,.$$

The translation of axes gives the equation

$$8x^2 - 4xy + 5y^2 - 36 = 0 \,,$$

while the rotation of axes through an angle θ such that $\tan 2\theta = \frac{B}{A - C}$, gives the transformed equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \,.$$

Primes have been omitted consistently in the interest of simplicity.

Exercises 8-10

Identify the graphs of the following equations. Obtain the transformed equation reduced to standard form. Sketch the graph, locating the center (if any) and indicate axes of symmetry.

1. $8x^2 - 12xy + 17y^2 - 20 = 0$
2. $3x^2 + 12xy - 13y^2 - 135 = 0$
3. $5x^2 - 6xy + 5y^2 - 16x + 16y + 8 = 0$
4. $9x^2 - 24xy + 16y^2 - 20x - 15y = 0$
5. $9x^2 - 24xy + 16y^2 + 60x - 80y + 100 = 0$
6. $3x^2 + 10xy + 3y^2 + 16x + 16y + 24 = 0$
7. $5x^2 + 6xy + 5y^2 - 16x - 16y + 8 = 0$
8. $27x^2 - 48xy + 13y^2 - 12x + 44y - 77 = 0$
9. $12x^2 - 7xy - 12y^2 - 41x + 38y + 22 = 0$
10. $13x^2 + 48xy + 27y^2 + 44x + 12y - 77 = 0$
11. $9x^2 - 24xy + 16y^2 + 90x - 120y + 200 = 0$
12. $10xy + 4x - 15y - 6 = 0$

Supplement to Chapter 10

GEOMETRIC TRANSFORMATIONS

S10-1. Isometries of the Line

In previous chapters we have seen examples of mappings of a line onto a line and of a plane onto a plane. Some of these had the property of preserving the distance between any two points and are therefore called "isometries," (from Greek, ισος meaning same and μετρειν meaning to measure). Therefore, an isometry, having this property, will map any configuration onto a congruent configuration. In fact this amounts to a definition of congruence. In this chapter we want to investigate the isometries of the line and of the plane and consider other types of mappings or transformations.

Let us consider in more generality the isometric transformations of a line. Each point P with coordinate x will be mapped onto its image point P' with coordinate $x' = f(x)$. Furthermore, for any two points with coordinates x_1 and x_2, we have

$$(1) \qquad |x_1 - x_2| = |f(x_1) - f(x_2)| .$$

We distinguish two cases according as the origin is a fixed point or is not a fixed point.

If zero is a fixed point, we have $f(0) = 0$, so that with $x_2 = 0$, (1) becomes

$$|x_1 - 0| = |f(x_1) - f(0)|$$

or

$$|x| = |f(x)| .$$

This implies that either $f(x) = x$ or $f(x) = -x$. In the former, each point is mapped onto itself and this is called the identity transformation I. In the latter we have a transformation which can be described as a reflection in the point 0, because each point is mapped onto its mirror-like image with respect to 0.

If zero is not a fixed point, it is mapped onto some point with a non-zero coordinate and we can write $f(0) = a \neq 0$. Thus with $x_2 = 0$, (1) becomes

$$|x_1 - 0| = |f(x_1) - f(0)|$$

or

$$|x| = |f(x) - a| .$$

This implies that either $f(x) - a = x$ or $f(x) - a = -x$. The former is $f(x) = x + a$ which is a translation and the latter is $f(x) = -x + a$. The transformation represented by $f(x) = -x + a$ can be described by saying that the image of any point is obtained by a reflection in the origin followed by a translation of a. We now have

THEOREM S10-1. An isometry of the line is either

(1) the identity transformation

(2) a translation

(3) a reflection in the origin

or (4) a reflection in the origin followed by a translation;

and conversely.

The fourth possibility in Theorem S10-1 raises the general question of one transformation followed by another. If the first transformation is f and the second is g, we define the product or composite transformation to be the transformation

$$gf : x \rightarrow x' = g[f(x)] ,$$

where $x \rightarrow x'$ means that the image of x under the mapping gf is x'. As we have seen, the transformation $x \rightarrow -x + a$ is a composite of $f(x) = -x$ followed by $g(x) = x + a$ since $g[f(x)] = -x + a$. From the definition of an isometry, it seems reasonable to expect that the product of two isometries should be an isometry. We show this to be true in the following case.

Example. Show that the translation $f(x) = x + a$ followed by the translation $g(x) = x + b$ is an isometry.

Solution. We have

$$g[f(x)] = (x + a) + b = x + (a + b)$$

which represents a translation. Thus the composite transformation is an isometry.

Exercises S10-1

1. By considering the remaining possibilities in similar fashion, show that the composite of any two isometries of the line is again an isometry.

2. Prove the converse of Theorem S10-1.

In the first exercise above, it was necessary to consider a translation followed by a reflection. If $g(x) = x + a$ is followed by $f(x) = -x$, the composite transformation is

$$fg : f[g(x)] = -(x + a) = -x - a .$$

This is certainly an isometry since it is a reflection followed by a translation $-a$. We see that composition of transformations is not necessarily commutative since in this case $fg \neq gf$. However we can generate any isometry by an appropriate sequence of compositions using only translations and reflections. It is not difficult to show that the isometries of a line form a group since the operation of composition is associative and to each isometry f, there exists an inverse isometry f^{-1} such that $f^{-1}f = I$. As we have observed, this group is non-commutative.

S10-2. Isometries of the Plane

In previous chapters we considered two changes of coordinate systems in the plane called translation and rotation. The same effect can be produced by mappings of the plane onto itself, which leave the coordinate axes unchanged. The contrast to this is the previous approach in which the plane remained fixed and the coordinate axes were changed.

In this context, a translation is a mapping of the form

$$(x,y) \to (x',y') = (x + h, y + k) .$$

A rotation is a mapping in which each point is mapped onto a point the same distance from the origin. These points determine rays from the origin which form an angle in standard position whose measure is increased by θ .

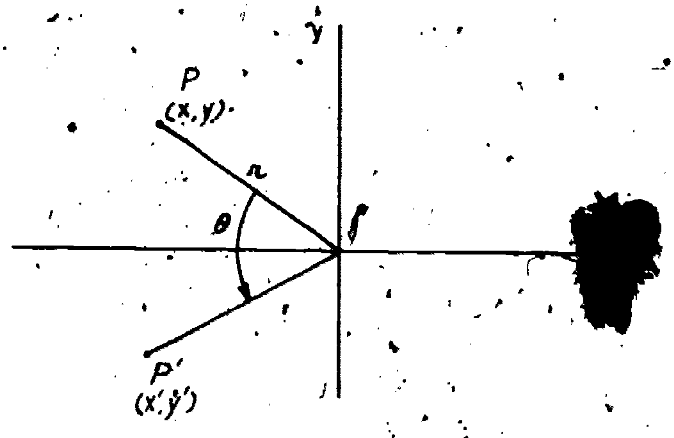


Figure S10-1

Let (r,ϕ) be a point P described in polar coordinates where the polar axis is the positive side of the x-axis. The rotation mapping can now be written as

$$(r,\phi) \to (r,\phi + \theta) .$$

In terms of rectangular coordinates, we have

$$\begin{aligned} x' &= r \cos (\phi + \theta) = r \cos \phi \cos \theta - r \sin \phi \sin \theta \\ &= x \cos \theta - y \sin \theta \\ y' &= r \sin (\phi + \theta) = r \sin \phi \cos \theta + r \cos \phi \sin \theta \\ &= x \sin \theta + y \cos \theta . \end{aligned}$$

The proofs that these mappings are isometries are left as exercises.

The previous discussion of reflection with respect to a point can be extended to the plane. A reflection in the origin can be defined by the transformation

$$(x,y) \to (x',y') = (-x,-y) .$$

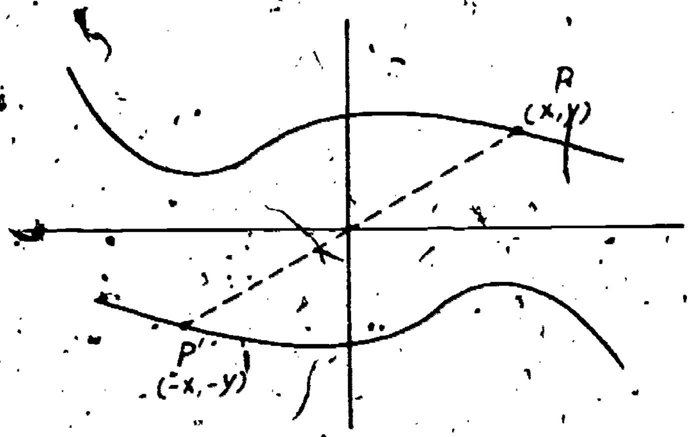


Figure 10-2

The description of this transformation is particularly simple in terms of polar coordinates since $P(r,\phi) \to P'(-r,\phi)$. By using the distance formula for the appropriate coordinate system, it is easy to verify that this transformation is an isometry of the plane. However a rotation of π radians is the same transformation. This can be seen by letting $\theta = \pi$ in the rectangular description of a rotation to obtain

$$x' = x \cos \pi - y \sin \pi = -x$$
$$y' = x \sin \pi + y \cos \pi = -y ,$$

or by letting $\theta = \pi$ in the polar description to obtain

$$(r,\phi) \to (r , \phi + \pi) .$$

The last ordered pair represents a point in polar coordinates which can also be represented as $(-r,\phi)$.

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We now introduce another transformation which can be described as a reflection in a line. The image of a point is found by constructing a perpendicular to the line and extending it on the other side a distance equal to the distance of the point from the line.

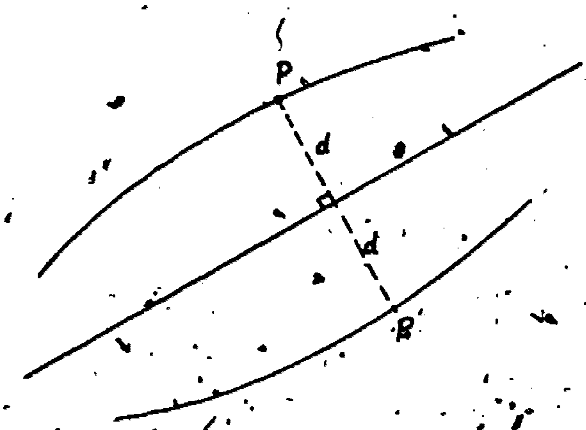


Figure S10-3

The transformation equations for reflections in certain lines can be written down immediately. For instance, for reflection in the x-axis, we have $(x,y) \longrightarrow (x',y') = (x,-y)$.

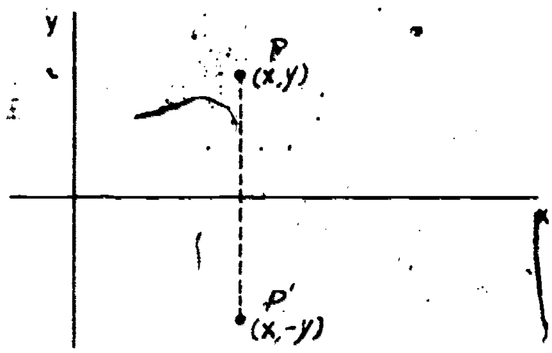


Figure S10-4

For reflection in the y-axis, we have

$$(x,y) \rightarrow (x',y') = (-x,y) .$$

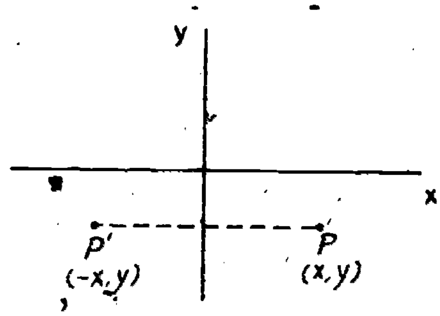


Figure S10-5

We can similarly define the product of two transformations of the plane onto itself, and again would expect the product of two isometries to be an isometry. In fact we will show that any isometry of the plane can be described solely in terms of reflections. Thus the group of isometries of the plane with composition can be generated from the set of reflections alone.

Example. Find the isometry composed of reflection in the line $x = 1$ followed by reflection in the line $x = 4$.

Solution.

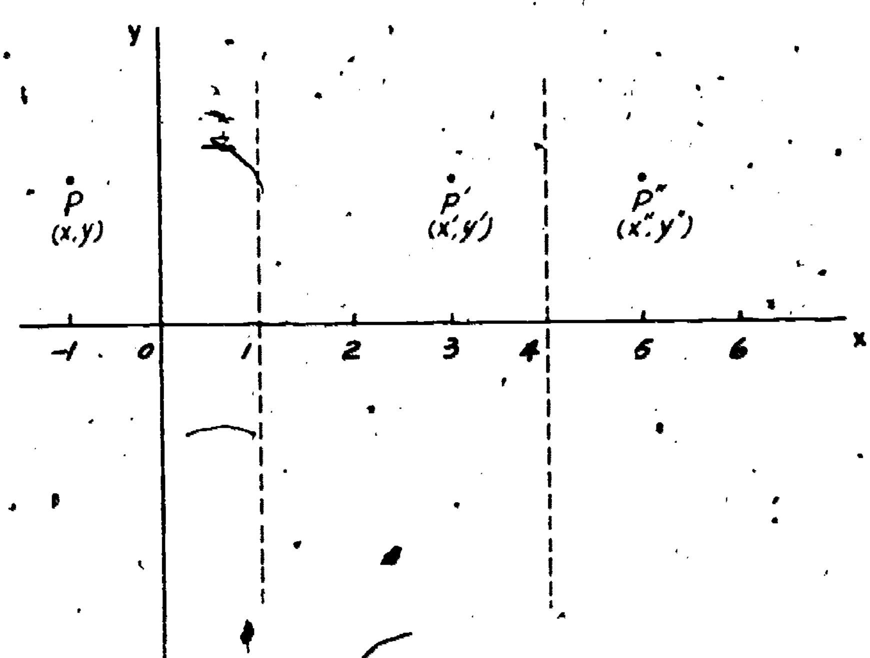


Figure S10-6

The first reflection maps $P(x,y)$ onto $P'(x',y') = (-x + 2, y)$ and the second maps (x',y') onto $(x'',y'') = (-x' + 8, y')$. By composition of the mappings, we immediately have

$$x'' = -x' + 8 = -(-x + 2) + 8 = x + 6$$
$$y'' = y' = y$$

and we recognize these as the equations of a translation which maps each point onto the point six units to the right.

Exercises S10-2

1. Do the two mappings in the example commute under composition?
2. Find the equations to describe the mapping of reflection in an arbitrary vertical line $x = h$ and in an arbitrary horizontal line, $y = k$.
3. Using Exercise 2 find the composite mapping given by successive reflection in either 2 horizontal or 2 vertical lines.
4. What is the composite mapping given by reflection in the line $x = h$ followed by reflection in the line $y = k$?
5. Do the mappings in Exercises 3 and 4 commute under composition?

S10-3. Reflections and Isometries

The above exercises illustrate the proposition that any translation or any reflection in a point can be obtained by a succession of reflections in appropriate lines. We observed previously that a reflection in O is equivalent to a rotation of π radians, so that a rotation of π radians can be obtained by a succession of reflections. Let us try to establish further connections between reflections and rotations by describing a reflection in a line L in terms of polar coordinates. Choose the pole of the coordinate system on the line in which the reflection is to be made and let the equation of the line L be $\theta = k$, a constant.

From Figure S10-7 it can be seen that $r' = r$ and that a measure of ϕ' is $\theta + (\theta - \phi) = 2\theta - \phi$ for this particular diagram. We can show this in general if we start with the angle 2θ and subtract the angle ϕ to arrive at the terminal side of the angle ϕ'. Thus the reflection in the line L is the mapping

(1) $$R_L : (r,\phi) \to (r',\phi') = (r, 2\theta - \phi)$$

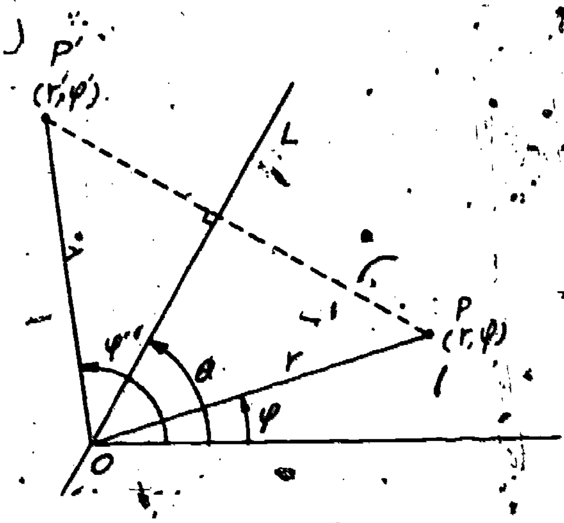


Figure S10-7

Suppose we now carry out successive reflections in two lines L and M through 0 with equations $\phi = \theta_1$, and $\phi = \theta_2$. By (1) we can denote the reflections by

$$R_L : (r,\phi) \rightarrow (r',\phi') = (r\ ,\ 2\theta_1 - \phi)$$

$$R_M : (r',\phi') \rightarrow (r'',\phi'') = (r'\ ,\ 2\theta_2 - \phi') .$$

The composite transformation R_L followed by R_M can be described as

$$R_M R_L : (r,\phi) \rightarrow (r'',\phi'')$$

where $$r'' = r' = r$$

and $$\phi'' = 2\theta_2 - \phi' = 2\theta_2 - (2\theta_1 - \phi) = \phi + 2(\theta_2 - \theta_1) .$$

We recognize this as the description of a rotation of $2(\theta_2 - \theta_1)$; thus, the composite mapping of two reflections in intersecting lines is a rotation.

<u>Exercises S10-3</u>

1. By reversing the above argument, prove that any rotation is the product of line reflections.
2. Using the notation of the preceding discussion, determine $R_L R_M$.

We are now in a position to prove

__THEOREM S10-2.__ Any isometry of the plane is composed of at most three line reflections.

__Proof.__ Assume we have some distance-preserving transformation which will therefore map an arbitrary triangle ABC onto a congruent triangle A'B'C'. The line through the points A and B may or may not intersect the line through the points A' and B'. Hence we consider two cases.

__Case I.__ The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect.

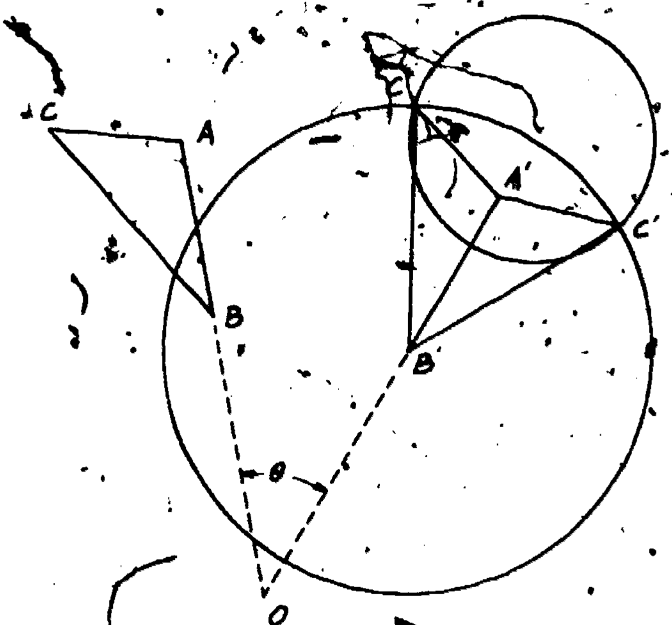


Figure S10-8

From Figure S10-8 we see there are two possible positions for the point C' at points of intersection of the circles given by the conditions $d(A',C') = d(A,C)$ and $d(B',C') = d(B,C)$. For one position of C', the transformation is a rotation θ about O, which can be represented as the product of two line reflections. For the other position of C', the transformation is the same rotation followed by a reflection in the line through A' and B', and therefore is the product of three line reflections.

Case II. The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel.

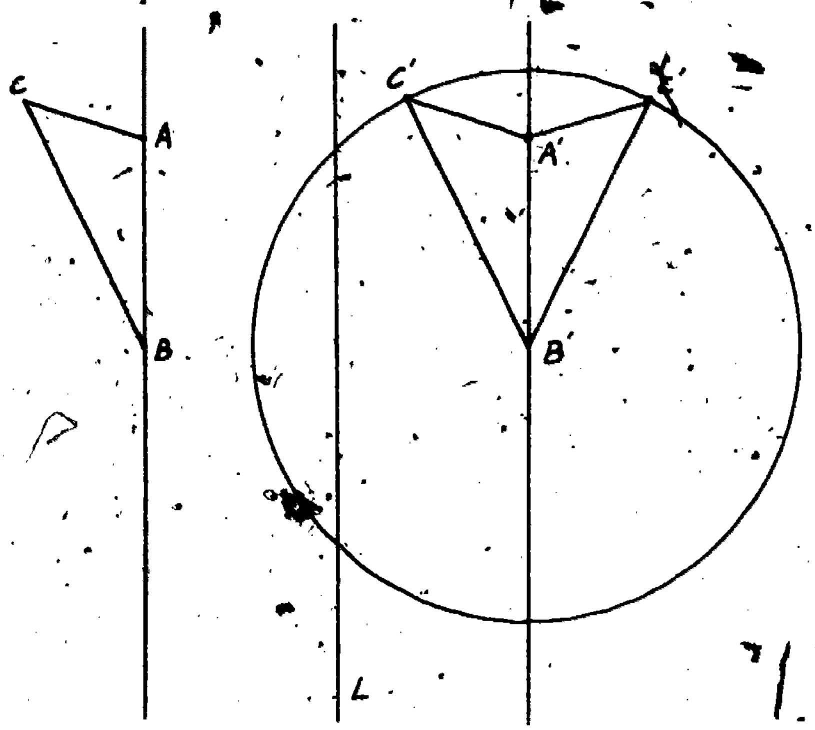


Figure S10-9

Once again there are two possible positions for the point C'. Consider the line L midway between the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$. Then for one position of C', the transformation is a reflection in L. For the other position of C', the transformation is a reflection in L followed by a reflection in the line $\overleftrightarrow{A'B'}$, which completes the proof of the theorem.

S10-4. Non-isometric Transformations

In Section S2-2, in addition to the transformations of a line onto itself called translation and reflection, the transformation's expansions and contractions were defined. An expansion is a mapping $x \to x' = ax$ where $a > 1$ and a contraction is a mapping $x \to x' = ax$ where $0 < a < 1$. It is apparent that neither of these is an isometry since the origin is mapped onto itself and the point whose coordinate is 1 is mapped onto the point

whose coördinate is a, but $|1 - 0| \neq |a - 0|$. We may consider the compositions of these transformations with themselves and with isometries to obtain a general class of transformations of the form

$$x \to x' = ax + b \quad a \neq 0 .$$

known as the class of linear transformations. As we noted in Section S2-2, this set of linear transformations with the operation of composition forms a group.

The idea of a linear transformation extends naturally to the plane by considering the mapping $(x,y) \to (x',y')$ where

$$x' = ax + by + h , \quad |a| + |b| \neq 0 ,$$
$$y' = cx + dy + k , \quad |c| + |d| \neq 0 ,$$

We see immediately that this mapping is the composition of the mapping

$$(x,y) \to (x',y') = (ax + by , cx + dy)$$

followed by the translation

$$(x',y') \to (x'',y'') = (x' + h , y' + k) .$$

Therefore we consider a subset of the set of linear transformations of the plane, namely those transformations of the form $(x,y) \to (x',y') = (ax + by , cx + dy)$ which leave the origin fixed. This subset includes the rotations and reflections in the plane previously discussed in this chapter. One of the things that can be done in general with this subset is to investigate whether it forms a group under composition. The identity mapping is an identity element for the operation of composition. Hence a given mapping will have an inverse if it can be followed by a mapping which will map (x',y') back onto (x,y). To find whether such a mapping exists, we consider the composite mapping $(x,y) \to (x',y') = (ax + by , cx + dy)$ followed by $(x',y') \to (x'',y'') = (px' + qy' , rx' + sy)$. We obtain the mapping $(x,y) \to (x'',y'')$ where

$$x'' = p(ax + by) + q(cx + dy) = (ap + cq)x + (bp + dq)y$$
$$y'' = r(ax + by) + s(cx + dy) = (ar + cs)x + (br + ds)y ,$$

which is a mapping of the same form. Thus, given a, b, c, d, we want to determine p, q, r, s so that the composite mapping is the identity mapping; that is, so that

$$\begin{aligned} ap + cq &= 1 \\ bp + dq &= 0 \\ ar + cs &= 0 \\ br + ds &= 1 . \end{aligned}$$

This is actually two linear systems, each consisting of two equations in two unknowns, which can be solved to obtain

$$p = \frac{d}{ac - bc}, \quad q = \frac{-b}{ad - bc}, \quad r = -\frac{c}{ad - bc}, \quad s = \frac{a}{ac - bc},$$

if $ad - bc \neq 0$. Thus, a mapping will have an inverse if and only if $ad - bc \neq 0$. It is left as an exercise to prove that this set of transformations is associative. We combine these results in

<u>THEOREM S10-3</u>. The set of linear transformations of the form

$$(x,y) \rightarrow (x',y') = (ax + by, cx + dy)$$

where $ad - bc \neq 0$, forms a group under the operation of composition.

We now consider examples of linear transformations which are not isometries.

<u>Example 1</u>. Discuss the linear transformation

$$(x,y) \rightarrow (x',y') = (2x + 3y, x - y).$$

<u>Discussion</u>. We start by examining what happens to points on certain lines under this transformation. For instance, a point on the x-axis, $(a,0)$, is mapped onto the point $(2a,a)$, which lies on the line $y = \frac{1}{2}x$. A point on the y-axis, $(0,a)$, is mapped onto the point $(3a,-a)$, which lies on the line $y = -\frac{1}{3}x$. If a point lies on a line whose equation is $ax + by + c = 0$, we can find a condition on the coordinates of its image by expressing x and y in terms of x' and y' and substituting in the equation. From the equations of the transformation we get

$$x = \frac{1}{5}(x' + 3y')$$

$$y = \frac{1}{5}(x' - 2y).$$

(This also shows that any point (x',y') is the image of some (x,y). Thus a point on the line is mapped onto a point (x',y') such that

$$a(x' + 3y') + b(x' - 2y') + 5c = 0$$

or

$$(a + b)x' + (3a - 2b)y' + 5c = 0.$$

which is an equation of a line. Hence a line is mapped onto a line, and if the line contains the origin (i.e., $c = 0$), so does its image. The images of other loci can be similarly determined.

Example 2. Discuss the linear transformation

$$(x,y) \rightarrow (x',y') = (x + y, 2x + 2y).$$

Discussion. We first observe that this transformation does not belong to the group described in Theorem S10-3 since $1 \cdot 2 - 1 \cdot 2 = 0$. Hence it does not possess an inverse mapping under composition. We investigate this transformation geometrically. A point (a,b) is mapped onto the point $(a + b, 2a + 2b)$. This image lies on the line $y = 2x$, so that the plane is mapped onto a single line in the plane. Furthermore, infinitely many points in the plane are mapped onto each point on the line $y = 2x$. Thus the mapping does not have an inverse mapping in the sense of assigning a unique pre-image to each image point.

Since there is a one-to-one correspondence between points in the plane and complex numbers, it is not surprising that mappings of the plane can be related to complex numbers. Recall that if we have a rectangular coordinate system, this correspondence is established by associating the point (a,b) and the complex number $a + bi$. Thus any of the mappings we have discussed so far can be considered as mappings of the set of complex numbers into itself. That is, if (x,y) is mapped onto (x',y'), we consider the complex number $x + yi$ mapped onto the complex number $x' + y'i$. Since functions are mappings, functions whose domain and range are the set of complex numbers give a mapping of the set of complex numbers into itself. For example consider the function defined by $f(z) = 2z$, or the mapping $z \rightarrow z' = 2z$, where $z = x + yi$ and $z' = x' + y'i$. This function maps $x + yi$ onto $2x + 2yi$, which corresponds to mapping the point (x,y) onto the point $(x',y') = (2x,2y)$. An investigation of this mapping is left for an exercise.

We give another example of this relationship.

Example 3. Discuss the mapping defined by the equation $f(z) = z^2$.

Discussion. From the equation we have

$$z' = x' + y'i = z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi.$$

Hence, in terms of coordinates the mapping is the non-linear transformation

$$x' = x^2 - y^2$$
$$y' = 2xy.$$

We see from these equations that the hyperbola $x^2 - y^2 = k$ is mapped onto the line $x' = k$ and the hyperbola $2xy = k$ is mapped onto the line $y' = k$. (It is convenient to think of the functions as a mapping of the z-plane, with x and y coordinates, into the z'-plane, with x' and y' coordinates.) We also have

$$x'^2 + y'^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = (x^2 + y^2)^2$$

so that the circle $x^2 + y^2 = r^2$ in the z-plane is mapped onto the circle $x'^2 + y'^2 = r^4$ in the z'-plane. We see that in trying to develop a geometric description of a mapping, it is sometimes more fruitful to discuss the images of certain loci rather than the images of individual points. This mapping is an example of an important class of functions of z known as <u>conformal mappings</u> which have the property of preserving the angle of intersection of two curves. This property is of fundamental importance in the general theory of functions of a complex variable z. In particular, polynomials in z and their quotients will provide conformal mappings.

Sometimes information about a mapping can be obtained by using the polar representation of a complex number. Thus, if θ is the angle in standard position which contains (x,y) on its terminal side, we can write

$$z = x + yi = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{x^2 + y^2}$. De Moivre's Theorem gives us

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta).$$

Thus, in the mapping $z \to z' = z^2$, the point (r,θ) is mapped onto the point $(r^2, 2\theta)$, which gives a general geometric description of the mapping.

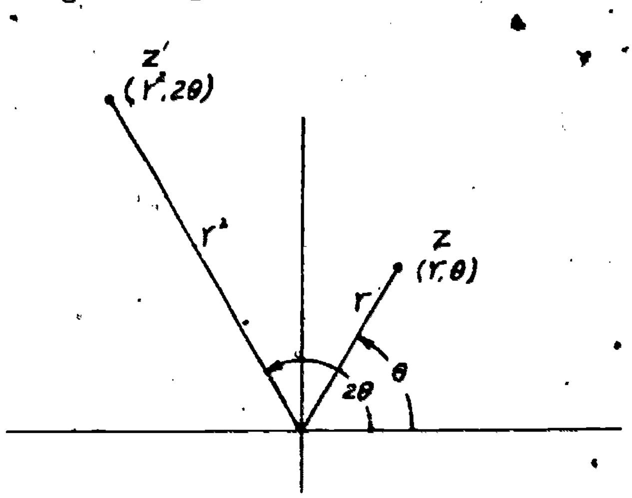


Figure S10-10

Exercises S10-4

1. Show that any transformation belonging to the group in Theorem S10-3 will map a line onto a line.

2. Discuss the transformations $(x,y) \to (2x,2y)$, $(x,y) \to (\frac{1}{2}x, \frac{1}{2}y)$, and $(x,y) \to (2x,3y)$ by finding the image of $x^2 + y^2 = 1$.

3. In Example 2, find those points which are mapped onto the same point on $y = 2x$.

4. Show that the angle between two lines through the origin is preserved under the mapping $z \to z' = kz$.

5. Discuss the mapping $z \to z' = \frac{1}{z}$.

6. Find various equations to represent the mapping called "inversion in a circle," in which a point at distance d from the origin is mapped onto the point at distance $\frac{1}{d}$ from the origin lying on the same ray from the origin. The origin is mapped onto itself.

7. Prove that the set of linear transformations $(x,y) \longrightarrow (x',y') = (ax + by, cx + dy)$ is associative.

S10-5. Matrix Representation of Transformations

In the previous section we saw that the product of two linear transformations is again a linear transformation. It is convenient to introduce a notation to represent a linear transformation

$$x' = ax + by$$
$$y' = cx + dy.$$

Since the coefficients of x and y determine the mapping, we represent the mapping by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a matrix in general is simply a rectangular array of numbers arranged in rows (horizontally) and columns (vertically). The composite mapping fg is the mapping g followed by the mapping f. Thus, as we saw in the previous section, the mapping whose matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

followed by the mapping whose matrix is

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

is the mapping whose matrix is

$$\begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{pmatrix}.$$

Hence, it is natural to define a binary operation on these matrices as follows.

DEFINITION. (Matrix multiplication)

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{pmatrix}.$$

Observe that each entry in the product matrix is the inner product of a row in the left factor by a column in the right factor. Because of this, matrix multiplication can be described as "row into column" multiplication.

Example 1. Find the matrix which represents a mapping described by rotation θ.

Solution. The equations of this mapping using rectangular coordinates are

$$x' = x \cos \theta - y \sin \theta$$
$$y' = x \sin \theta + y \cos \theta .$$

The corresponding matrix is

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Example 2. Using matrices find the mapping composed of a reflection in the x-axis followed by a reflection in the y-axis.

Solution. As we have seen, the equations for a reflection R_x in the x-axis are

$$x' = x = 1 \cdot x + 0 \cdot y$$
$$y' = -y = 0 \cdot x + (-1)y$$

so that the corresponding matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The equations for a reflection R_y in the y-axis are

$$x' = -x = (-1)x + 0 \cdot y$$
$$y' = y = 0 \cdot x + 1 \cdot y$$

with matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix for the composite mapping R_yR_x, is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which corresponds to the mapping $(x,y) \rightarrow (-x,-y)$. This, as we have seen, is a reflection in O or a rotation of π radians.

Exercises S10-5a

(Use Matrices)

1. Using the notation of the example above, find R_xR_y.

2. Find the matrix for

   (a) reflection in the line $y = x$.

   (b) reflection in the line $y = -x$,

3. Find the matrix for, and interpret geometrically, a reflection in the line $y = x$ followed by a rotation of $\frac{\pi}{2}$ radians.

4. Describe the mapping which results from a rotation θ_1 followed by a rotation θ_2.

5. Show that matrix multiplication is associative but not commutative.

*6. Show that the matrix for a reflection in a line through O with inclination θ is

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

(Hint: While this can be done directly in rectangular coordinates using trigonometry, it is also interesting to solve the problem using polar coordinates.) Verify that this matrix includes the previously discussed cases $\theta = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, and $\frac{3\pi}{4}$ radians.

7. Find the matrix for a reflection in a line through 0 with inclination θ_1 followed by a reflection in a line through 0 with inclination θ_2. Show that the answer agrees with previous results.

We have a one-to-one correspondence between two-by-two matrices (2 rows and 2 columns) and linear transformations of the plane which leave the origin unchanged. We also see, by the definition of matrix multiplication, that the product of two matrices corresponds to the mapping composed of the mappings corresponding to the matrices. Thus, the two systems are isomorphic in the sense that any operations on mappings can also be interpreted in terms of operations on the corresponding matrices. Hence Theorem S10-3 has an analogue for matrices as follows.

<u>THEOREM</u> <u>S10-4</u>. The set of matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $ad - bc \neq 0$, forms a group under the operation of matrix multiplication.

The number $ad - bc$ is called the <u>determinant</u> <u>of</u> <u>the</u> <u>matrix</u> and the matrix is called <u>non-singular</u> if $ad - bc \neq 0$. Thus the set in the theorem is the set of non-singular two-by-two matrices.

Since we found the inverse of a mapping in the proof of Theorem S10-3, we may write the inverse of a non-singular matrix under matrix multiplication as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{pmatrix}$$

We now consider the matrices of isometries of the plane which leave the origin fixed. By Theorem S10-2, any such matrix is the product of at most three matrices each of which represents a reflection. By Problem 6 in Exercises S10-5a, the matrix of a reflection in a line through the origin can be written

as

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

for some α. By Problem 7 in the same set of exercises, the matrix for the product of two line reflections, which is a rotation, can be written as

$$\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

for some β. Since matrix multiplication is associative, the matrix for the product of three line reflections can be written as a reflection matrix times a rotation matrix; that is, as

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

for appropriate α and β. This product is

$$\begin{pmatrix} \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta & \sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta \\ \sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta & -\cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha-\beta) & \sin(\alpha-\beta) \\ \sin(\alpha-\beta) & -\cos(\alpha-\beta) \end{pmatrix}.$$

Thus we have the following theorem.

<u>THEOREM</u> <u>S10-5</u>. Any isometry of the plane with O as a fixed point can be represented by one of the matrices

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \text{ or } \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

for suitable α.

<u>Corollary</u> <u>S10-5-1</u>. The determinant of a matrix which represents an isometry of the plane with O as a fixed point is either 1 or -1.

Let S be the set of matrices which can be written in either of the forms

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \text{ or } \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

for some α. We define two matrices to be equal if and only if they are identical, that is, if and only if their corresponding entries are equal. Thus the same matrix may arise from different values of α, but we consider

the matrices themselves and not the values of α. As we have seen, each such matrix represents an isometry (either a line reflection or a rotation), and by Theorem S10-5, any isometry with O a fixed point can be represented by such a matrix.

The set s forms a group, under the operation of matrix multiplication, which is a subgroup of the group described in Theorem S10-4.

Exercises S10-5b

1. Prove that the set s, just described, is a group.
2. Show that the determinant of the product of two square matrices of order 2^2 equals the product of their determinants.
3. Show that there exist matrices with determinant 1 or -1 which do not represent isometries.
4. Prove, using the distance formula, that an isometry with O as a fixed point has a matrix whose determinant is 1 or -1.
5. Any matrix in the set s, in addition to having determinant ± 1, has the property that the sum of the squares of the elements in any row or in any column is 1. Prove that if a matrix has determinant ± 1 and has the sum of the squares of elements in each column (or in each row) equal to 1, then it is a member of s.

S10-6. Symmetry

The symmetries of a geometric figure can be interpreted very nicely in terms of mappings. If a figure is mapped into itself by a particular isometry, then it has the particular kind of symmetry described by the isometry. Thus a figure may have symmetry with respect to a point if it is mapped into itself upon reflection in that point; it may have symmetry with respect to a line if it is mapped into itself upon reflection in that line. The algebraic tests for symmetry arise from the equations of the various mappings.

As you have seen, it is possible to simplify the equations of various loci by using appropriate transformations. In particular it is possible to eliminate by translation the terms involving x and y in an equation for an ellipse or a hyperbola. Then a suitable rotation will eliminate the xy term. Geometrically, what this last step involves is the determination of a suitable rotation so that the x and y axes become axes of symmetry of the figure.

We now want to solve this problem by means of the algebra of matrices. We assume that a suitable translation has been made so that the hyperbola or ellipse has its center at the origin of a rectangular coordinate system. Hence its equation is

$$f(x,y) = Ax^2 + Bxy + Cy^2 = D \quad (1)$$

We want to determine a rotation so that the points which satisfy $f(x,y) = D$ will be mapped onto points which satisfy some equation not having an xy term. Since the constant term is unaffected by a rotation, we consider only the quadratic portion $f(x,y)$ of (1). If we extend our notion of matrix multiplication slightly, we can get a matrix representation of $f(x,y)$. We introduce matrices with one row and two columns or two rows and one column and define a product of one of these times a two-by-two matrix.

DEFINITION.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$(x \quad y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

$$(p \quad q)\begin{pmatrix} r \\ s \end{pmatrix} = (pr + qs)$$

Notice that the one-by-two (or two-by-one) matrices must occur in the proper position but that the multiplication is still row into column multiplication. We now associate with $f(x,y)$ the matrix

$$\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}$$

and verify without difficulty that

$$f(x,y) = (x \quad y)\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (2)$$

We can similarly express a rotation θ as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \quad (3)$$

By Theorem S10-4 this rotation matrix has an inverse. The Equation (3) is a statement of equality of matrices and hence each member can be multiplied on the left by the inverse matrix to obtain equal matrices.

$$(4) \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(This assumes associativity of a matrix product involving non-square matrices and the proof of this is left as an exercise.)

From the definition it is not hard to see that if

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

then

$$(ax + by \quad cx + dy) = (x \quad y)\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Thus from (4) we have

$$(x \quad y) = (x' \quad y')\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Substituting (4) and (5) into (2) we see that the rotation will transform $f(x,y)$ into

$$g(x',y') = (x'y')\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= (x'y')\begin{pmatrix} A\cos\theta - \frac{B}{2}\sin\theta & \frac{B}{2}\cos\theta - C\sin\theta \\ A\sin\theta + \frac{B}{2}\cos\theta & \frac{B}{2}\sin\theta + C\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= (x'y')\begin{pmatrix} A' & \frac{B'}{2} \\ \frac{B'}{2} & C' \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = A'x'^2 + B'x'y' + C'y'^2 .$$

We now want to determine θ so that $g(x',y')$ remains unchanged when x' is replaced by $-x'$ and also when y' is replaced by $-y'$. This will occur if $g(x',y')$ does not have an $x'y'$ term.

The coefficient of $x'y'$ in $g(x',y')$ is

$$B' = 2\{(A - C)\sin\theta \cos\theta - \frac{B}{2}(\sin^2\theta - \cos^2\theta)\}$$

$$= (A - C)\sin 2\theta + B\cos 2\theta .$$

Thus $\qquad B' = 0$ if

$$\theta = \frac{\pi}{4} \text{ radians, when } A = C$$

$$\theta = \frac{1}{2} \text{ arc tan } \frac{B}{C - A}, \text{ when } A \neq C .$$

In Chapter S7 the latter angle θ was $\frac{1}{2}$ arc tan $\frac{B}{(A - C)}$, since there, the axes were rotated, whereas in this treatment, the axes remain fixed and the plane is mapped onto itself. The calculations here do not differ from those in Chapter S7, but it is of interest to see them carried out in a different framework.

We may also use this approach to prove that the determinant of $f(x,y)$, which is $AC - \frac{B^2}{4}$, is invariant not only under a rotation, but also under any isometry which leaves O fixed. For this we use the fact that

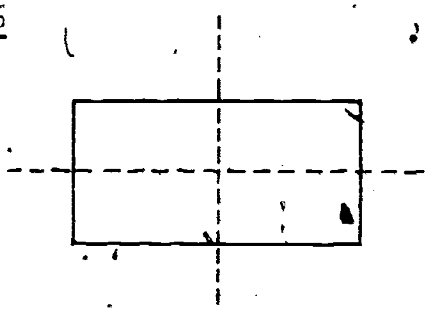
$$\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right| = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| \left|\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right| ,$$

which was shown in Problem 2, Exercises S10-5b. Thus if M is the matrix of such an isometry, we have

$$A'C' - \frac{B'^2}{4} = \left|\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| \cdot \left|\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}\right| \cdot \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$

$$= AC - \frac{B^2}{4} \quad \text{since} \quad ad - bc = \pm 1 .$$

Exercises S10-6



1. Describe (in terms of reflections alone) the isometries of the plane which in addition carry the outline of a given rectangle into itself,

I-1. reflection in y axis : $(x,y) \rightarrow (-x,y)$

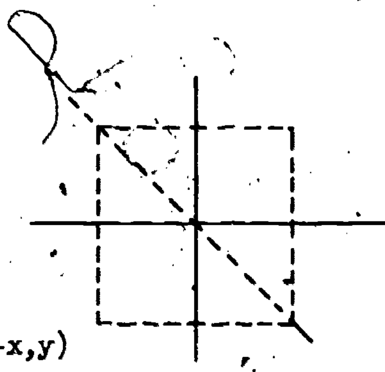
I-2. reflection in x axis : $(x,y) \rightarrow (x,-y)$

I-3. reflection in the origin $(x,y) \rightarrow (-x,-y)$

I-4. identity $(x,y) \rightarrow (x,y)$

I-1 followed I-2 has the same result as I-3.

2. Describe (in terms of reflections alone) the isometries of the plane which in addition carry the outline of a given square into itself,



I-1. reflection in the x-axis $(x,y) \longrightarrow (-x,y)$

I-2. reflection in the y-axis $(x,y) \longrightarrow (x,-y)$

I-3. reflection in the origin $(x,y) \longrightarrow (-x,-y)$

identity $(x,y) \longrightarrow (x,y)$

and in addition

I-5. reflection in the 45° line $(x,y) \longrightarrow (y,x)$

I-6. reflection in the 135° line $(x,y) \longrightarrow (-y,-x)$

3. Describe the isometries of 3-space which in addition carry a given cube into itself.